CS7800: Advanced Algorithms

Class 23: Randomized Algorithms V

- Global minimum cuts

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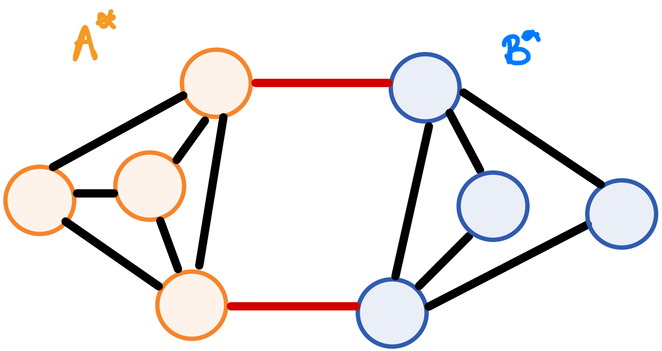
Global Minimum Cuts

Input: An undirected, unweighted graph $G=(V,E)$

Output: A partition (A,B) with $|A|,|B| \geq 1$
minimizing $|Cut(A,B)| = |\{(u,v) \in E : u \in A, v \in B\}|$

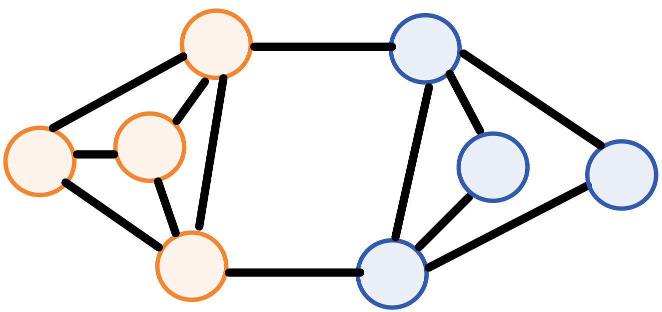
Compare to Min s-t Cut?



$|Cut(A^*,B^*)| = 2$

Finding Global Minimum Cuts

Reduction to Minimum s-t Cut



The Contraction Algorithm

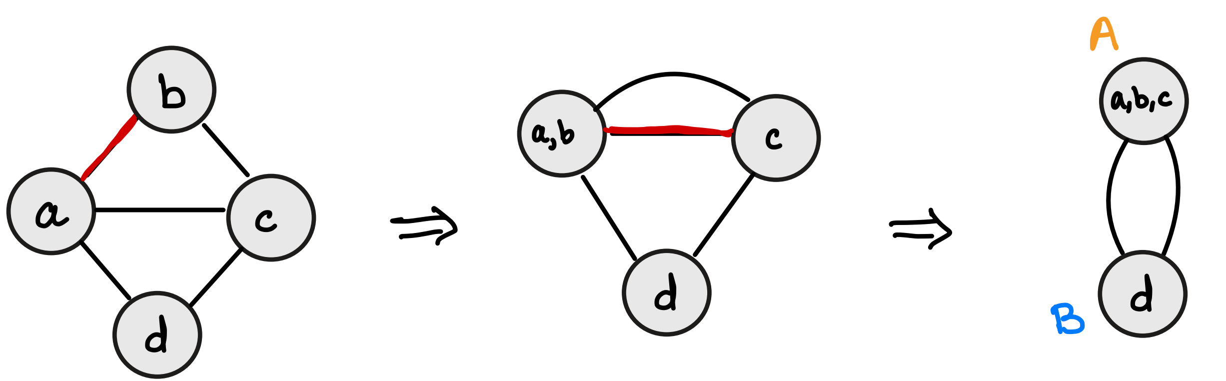
Contraction(G): ← Informal Version

While G has ≥ 3 nodes:

Choose a random $e = (u, v)$ in G

Contract u and v into a single supernode $\{u, v\}$

Return the contents of the two supernodes left in G



The Contraction Algorithm

Contraction$(G = (V, E))$:

For each $v \in V$: let $S(v) = \{v\}$

While G has $\geqslant 3$ nodes:

Choose an edge $(u, v) \in G$ uniformly

Replace u and v with new node $z_{u,v}$

$S(z_{uv}) = S(u) \cup S(v)$

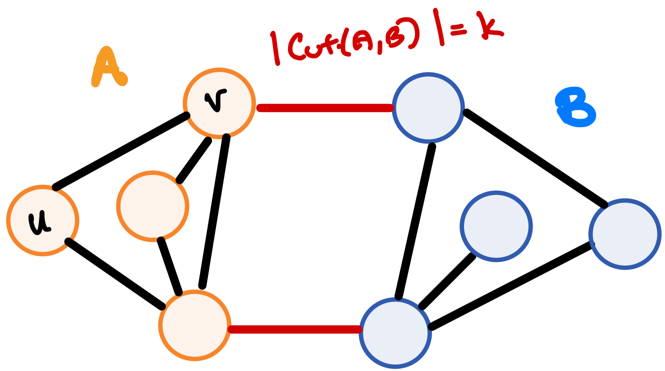
For the two nodes $z_1, z_2 \in G$ return $S(z_1), S(z_2)$

Can implement the contraction algorithm in time $O(n^2)$

Analyzing the Contraction Algorithm

Fix any global minimum cut (A, B), let $k = |\text{Cut}(A,B)|$

When does the contraction algorithm return (A,B)?



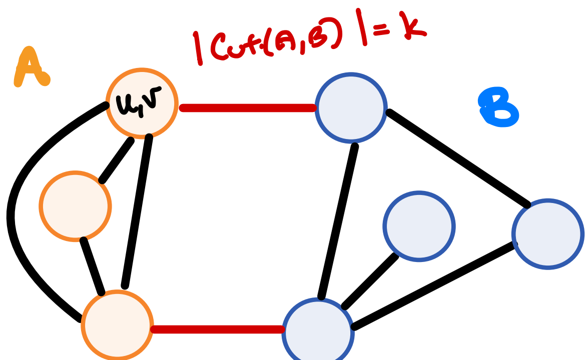
Analyzing the Contraction Algorithm

Fix any global minimum cut (A, B), let $k = |Cut(A,B)|$

Let E_i be the event that (A,B) is preserved after i contractions

$$\mathbb{P}(E_1) \geq 1 - \frac{k}{\frac{1}{2}kn} = 1 - \frac{2}{n} = \frac{n-2}{n}$$

$$\mathbb{P}(E_i \mid E_1 \cup \ldots \cup E_{i-1}) \geq$$



Analyzing the Contraction Algorithm

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$$\mathbb{P}(E_1) \geq 1 - \frac{k}{\frac{1}{2}kn} = 1 - \frac{2}{n} = \frac{n-2}{n}$$

$$\mathbb{P}(E_i \mid E_1 \cup \dots \cup E_{i-1}) \geq 1 - \frac{k}{\frac{1}{2}k(n-i+1)} = 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}$$

$$\mathbb{P}(A,B \text{ returned}) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2 \mid E_1) \cdot \dots \cdot \mathbb{P}(E_{n-2} \mid E_1 \cup \dots \cup E_{n-3})$$

$$\geq \left(\frac{n-2}{n}\right) \cdot \left(\frac{n-3}{n-1}\right) \cdot \left(\frac{n-4}{n-2}\right) \cdots \left(\frac{3}{5}\right) \cdot \left(\frac{2}{4}\right) \cdot \left(\frac{1}{3}\right)$$

Putting it Together

$GMC(G)$:

For $t = 1, \ldots, \binom{n}{2}\ln(n)$:

Let $(A^t, B^t) = \text{Contract}(G)$

Return the best cut you found

Independent Choices

Running Time: $O(n^2 \ln(n)) \times O(n^2) = O(n^4 \ln(n))$

Probability of Success:

Structure of Global Minimum Cuts

Fix any global minimum cut (A,B), let $k = |\text{Cut}(A,B)|$

$$\mathbb{P}(A,B \text{ returned}) \geq 1/\binom{n}{2}$$

Let $(A^1,B^1) \ldots (A^R,B^R)$ be all the min cuts

$$1 \geq \mathbb{P}(\text{A min cut is returned}) = \sum_{r=1}^{R} \mathbb{P}(A^r,B^r \text{ is returned}) \geq R/\binom{n}{2}$$

$$\implies \#\text{ of min cuts is } \leq \binom{n}{2}$$

Improving the Running Time

Fix any global minimum cut (A,B), let $k = |\mathrm{Cut}(A,B)|$

Let E_i be the event that (A,B) is preserved after i contractions

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$$\mathbb{P}(A,B \text{ returned}) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2 \mid E_1) \cdot \ldots \cdot \mathbb{P}(E_{n-2} \mid E_1 \cup \ldots \cup E_{n-3})$$

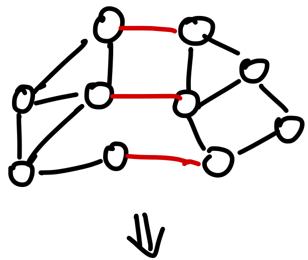
$$\geq \left(\frac{n-2}{n}\right) \cdot \left(\frac{n-3}{n-1}\right) \cdot \left(\frac{n-4}{n-2}\right) \cdots \left(\frac{3}{5}\right) \cdot \left(\frac{2}{4}\right) \cdot \left(\frac{1}{3}\right)$$

First contraction very rarely fails

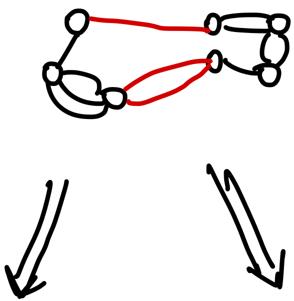
Last contraction often fails

Improving the Running Time

Want to make better use of our random trials



n nodes



$\frac{n}{2}$ nodes

$\mathbb{P}(\text{still OK})$

$$\geq \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\cdots\left(\frac{n/2-2}{n/2}\right)$$

$$\geq \frac{(n/2-1)(n/2-2)}{n(n-1)} \approx \frac{1}{4}$$

Run 2 indep copies

$\frac{n}{4}$ nodes

Keep recursively branching

SUCCEED

FAIL