CS7800: Advanced Algorithms

Class 20: Randomized Algorithms 1

- Probability Recap
- Load Balancing / Balls and Bins

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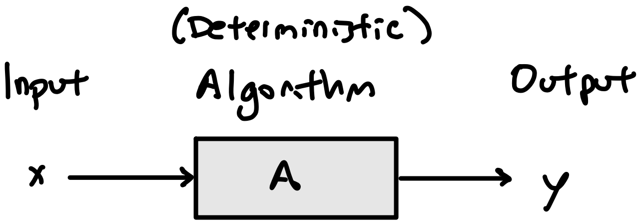
Randomness in Algorithms



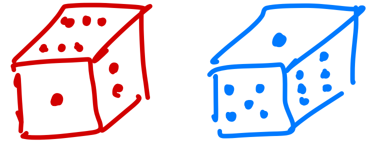
Correctness: For every x, $y = A(x)$ is correct

Running Time: For every x, $A(x)$ always runs in time $T(|x|)$

Kinds of Randomness:

(Discrete) Probability Toolkit

- Outcomes $\omega \in \Omega$



e.g $\Omega = \{1,2,3,4,5,6\}^2$

$\omega = (6, 1)$

- Probability $\mathbb{P} : \Omega \to \mathbb{R}$

  ① $\mathbb{P}(\omega) \geq 0$ ② $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

e.g $\mathbb{P}(\omega) = \frac{1}{36}$

- Events $E \subseteq \Omega$

e.g. $E = \{\omega : \omega_1 + \omega_2 = 7\}$

- Probability of an event is $\mathbb{P}(E) = \sum_{\omega \in \Omega} \mathbb{P}(\omega)$

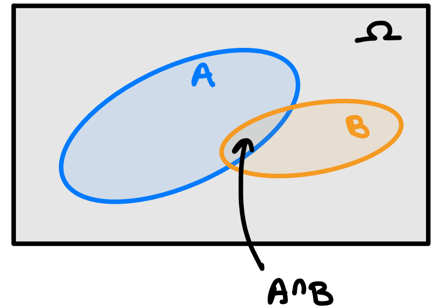
e.g. $\mathbb{P}(E) = 6 \times \frac{1}{36} = \frac{1}{6}$

- Can take complements ("not"), unions ("or"), intersections ("and")

Conditional Probability and Independence

- Conditional probability

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

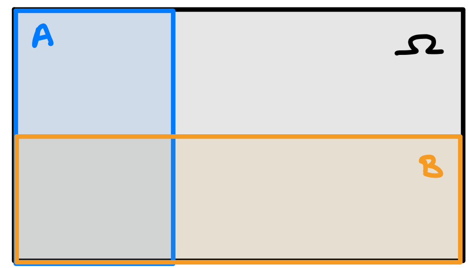


- Independence

A and B are independent

$$\Leftrightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\Leftrightarrow \mathbb{P}(A \mid B) = \mathbb{P}(A)$$



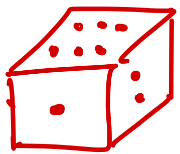
Ex. $A = \{\omega_1 = 3\}$
$B = \{\omega_2 = 3\}$ Independent

Ex. $A = \{\omega_1 = 3\}$
$B = \{\omega_1 \text{ is odd}\}$ Not independent

Random Variables (neither random nor variable)

- A random variable maps an <u>outcome</u> to a <u>value</u>

$\Omega = \{1,2,3,4,5,6\}$



$\xrightarrow{X(\omega) = \omega^3 + 1}$ 217

$\xrightarrow{X(\omega) = \text{King Henry the } \omega^{th}}$

- We treat integer-valued random variables as variables with unknown value

$$\mathbb{P}(X = k) = \mathbb{P}(\{\omega : X(\omega) = k\})$$
$$\mathbb{P}(X^2 \leq k) = \mathbb{P}(\{\omega : X(\omega)^2 \leq k\})$$

Expected Value

- The expected value of an integer random variable X is
  "average" "mean"

$$\mathbb{E}(X) = \sum_{k=-\infty}^{\infty} k \cdot \mathbb{P}(X=k)$$

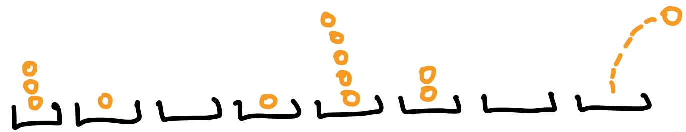
★ Expectation is linear $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$

- X and Y are independent if $\mathbb{P}(X=k \cap Y=\ell) = \mathbb{P}(X=k)\,\mathbb{P}(Y=\ell)$

- If X and Y are independent then $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$

Probability Case Study: Balls and Bins

Throw m balls into n bins independently



$$\Omega = \{1, \dots, n\}^m$$

$$\mathbb{P}(\omega_1, \omega_2, \dots, \omega_m) = \frac{1}{n^m}$$

$$\omega = (6, 8, 11, 2, 37, \dots)$$

Ball 1
Bin 6

Ball 2
Bin 8

Questions:

1. How long until bin 1 gets a ball?
2. How long until no bin is empty?
3. What is the maximum number of balls in any bin?

Waiting Time

How long until bin 1 gets a ball?

Waiting Time

Coupon Collector

How long until no empty bins?

Coupon Collector

Maximum Load