CS7800: Advanced Algorithms

Class 10: Linear Programming II

- LP Duality
- Minimax Theorem

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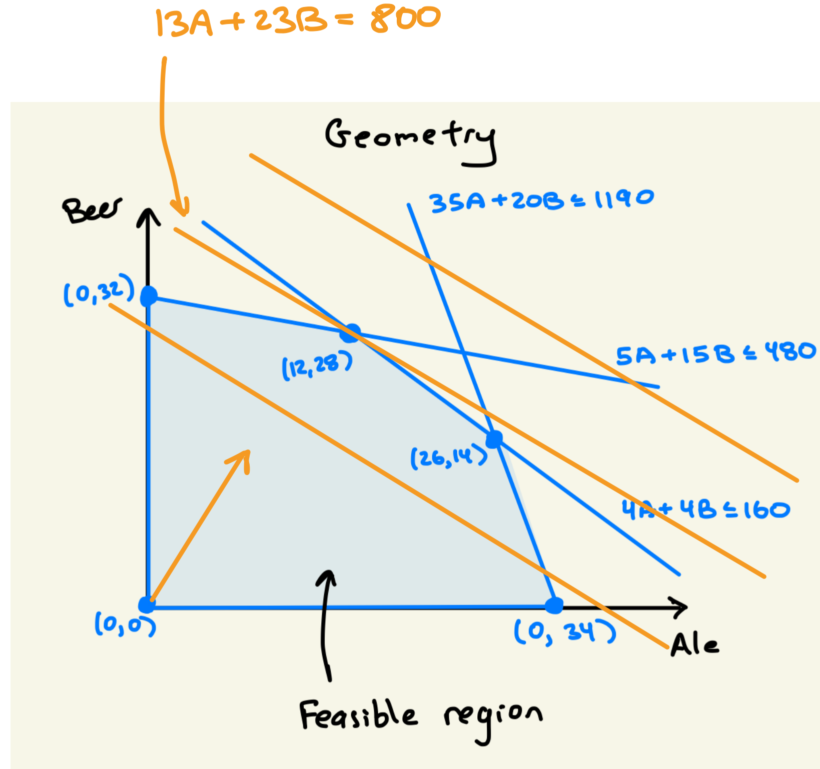
Linear Programming

$$
\begin{aligned}
\max_{A,B} \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \le 480 \\
& 4A + 4B \le 160 \\
& 35A + 20B \le 1190 \\
& A, B \ge 0
\end{aligned}
$$

optimal solution: $A = 12$, $B = 28$

optimal value: 800



How do we know we found an optimal solution?

Upper bound on optimal value

$$\max_{A,B} \; 13A + 23B$$

s.t.

$$5A + 15B \le 480 \xrightarrow{3\times} 15A + 45B \le 1440$$

$$4A + 4B \le 160 \xrightarrow{6\times} 24A + 24B \le 960$$

$$35A + 20B \le 1190$$

$$A, B \ge 0$$

optimal solution: $A = 12$, $B = 28$

optimal value: 800

How do we know we found an optimal solution?

Upper bound on optimal value

$$\max_{A,B} \quad 13A + 23B$$
$$\text{s.t.} \quad 5A + 15B \le 480$$
$$4A + 4B \le 160$$
$$35A + 20B \le 1190$$
$$A, B \ge 0$$

$1\times$

$2\times$

$$
\begin{array}{rl}
 & 5A + 15B \le 480 \\
(+) & 8A + 8B \le 320 \\
\hline
 & 13A + 23B \le 800
\end{array}
$$

optimal solution: $A = 12$, $B = 28$

optimal value: 800

<u>New Problem</u>: Derive the smallest upper bound on optimal value by combining constraints

- Coefficient on each constraint ≥ 0
- Combination upper bounds the objective

How the Dual of a Linear Program optimal solution?

primal (P)

optimization problem

$$\max_{A,B} \quad 13A + 23B$$

$$\text{s.t.} \quad 5A + 15B \le 480$$

$$4A + 4B \le 160$$

$$35A + 20B \le 1190$$

$$A, B \ge 0$$

optimal solution: $A = 12, B = 28$

optimal value: 800

dual (D)

optimization problem

The Dual of a Linear Program

primal (P) optimization problem

$$\boxed{\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \quad (y \in \mathbb{R}^m) \\ & x \ge 0 \end{aligned}}$$

$c \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

dual (D) optimization problem

$$\boxed{\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & y^T b \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}}$$

<u>Weak Duality</u>

For any feasible $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$c^T x \le y^T A x \le y^T b$$

The Dual of a Linear Program

Fact: The dual of the dual is the primal

Fact: Can take the dual without converting to standard form.

Primal	maximize	minimize	Dual
constraints	$a_i x = b_i$ $a_i x \le b_i$ $a_i x \geq b_i$	y_i unrestricted $y_i \geq 0$ $y_i \le 0$	variables
variables	$x_i \geq 0$ $x_i \le 0$ x_i unrestricted	$a_i y \geq c_i$ $a_i y \le c_i$ $a_i y = c_i$	constraints

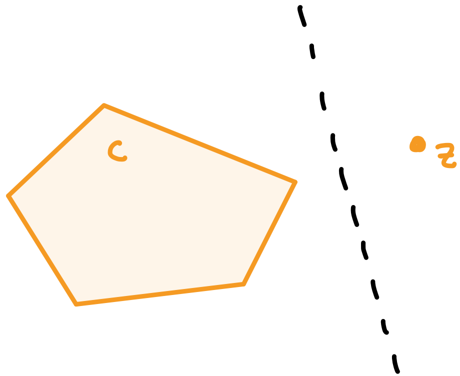
Strong LP Duality

Theorem: If the primal and dual are both feasible then they have the same optimal value

Special cases:

1. If the dual is infeasible, the primal is unbounded
2. If the dual is unbounded, the primal is infeasible

Strong Duality Proof Overview (Idea #1)



Separating Hyperplane Theorem: If $C \subseteq \mathbb{R}^n$ is a closed convex set and $z \in \mathbb{R}^n$ is any point not in C, there exists $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ s.t.

1. $\alpha^T x \geq \beta$ for all $x \in C$
2. $\alpha^T z < \beta$

Strong Duality Proof Overview (Idea #2)

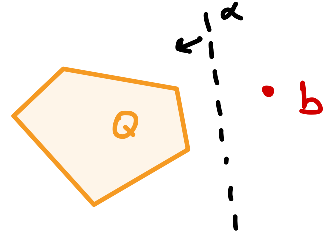
Farkas' Lemma: Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following is true:

1. There exists $x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$
2. There exists $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ and $y^T b < 0$

Strong Duality Proof Overview (Idea #2)

> **Farkas' Lemma:** Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following is true:
>
> ① There exists $x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$
>
> ② There exists $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ and $y^T b < 0$

Proof Sketch:



- Let $Q = \{ w : \exists x \geq 0 \;\; Ax = w \}$
- Assume !① so $b \notin Q$
- SHT $\Rightarrow$ $\exists$ $\alpha^T w \geq 0$ for every $w \in Q$ and $\alpha^T b < 0$

  small trick here

  easy part
- Claim: setting $y = \alpha$ satisfies $y^T A \geq 0$ and $y^T b < 0$
  - $(\alpha^T A)_j = \alpha^T \cdot (j^{th}$ column of $A)$
  - $(j^{th}$ column of $A) \in Q$ $\Rightarrow (\alpha^T A)_j \geq 0$ for all $j = 1, \ldots, n$

Strong Duality Proof Overview (Idea #3)

(P) $\max_{x} \; c^T x$

s.t. $Ax = b$

$x \geq 0$

optimal value is v^*

Application: The Minimax Theorem

Zero-Sum Games:

- Two players Rowena and Colin
- Rowena chooses an action in $[m]$ Colin chooses in $[n]$
- Payoffs $A \in \mathbb{R}^{m \times n}$

$$\left.\begin{array}{l}\text{Rowena plays } i \\ \text{Colin plays } j\end{array}\right\} \Longrightarrow \underbrace{\begin{array}{l}\text{Rowena gets } A_{ij} \\ \text{Colin gets } -A_{ij}\end{array}}_{\text{"zero-sum"}}$$

- Players can play randomly

$$\left.\begin{array}{lll}\text{Rowena: } r = (r_1, \dots, r_m) & \sum_i r_i = 1 & r_i \geq 0 \\ \text{Colin: } c = (c_1, \dots c_n) & \sum_j c_j = 1 & c_j \geq 0\end{array}\right\} \Longrightarrow$$

Rowena's expected payoff is

$$\sum_{ij} r_i c_j A_{ij} = r^T A c$$

Application: Minimax Thm

How would Rowena play if she went first?

How would Colin play if he went first?

Minimax Theorem:

Zero-Sum Games:

- Two players Rowena and Colin
- Rowena chooses an action in $[m]$ Colin chooses in $[n]$
- Payoffs $A \in \mathbb{R}^{m \times n}$

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Rowena's expected payoff is

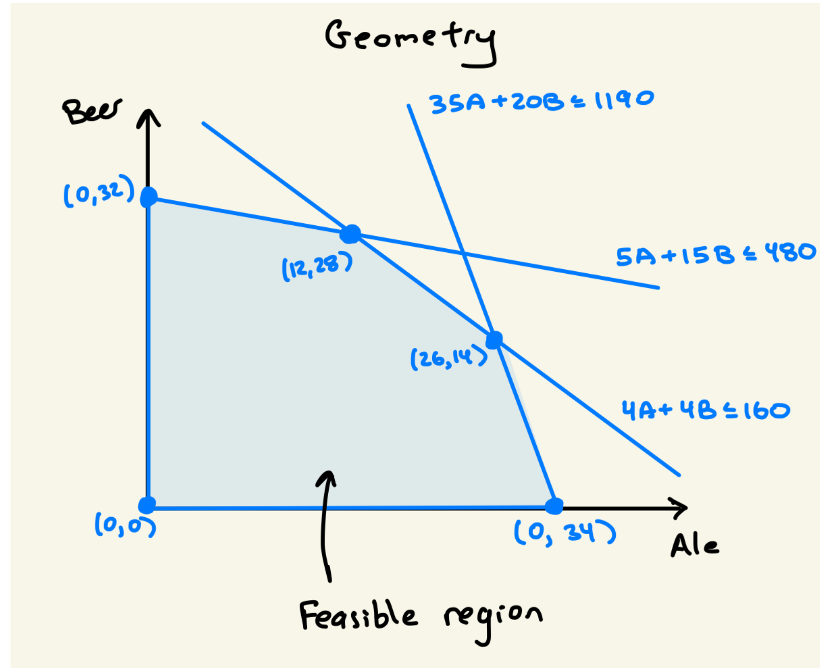
$$\sum_{ij} r_i c_j A_{ij} = r^T A c$$

Application: Minimax Thm Proof

Solving Linear Programs: Simplex

Basic Feasible Solutions (Geometry)

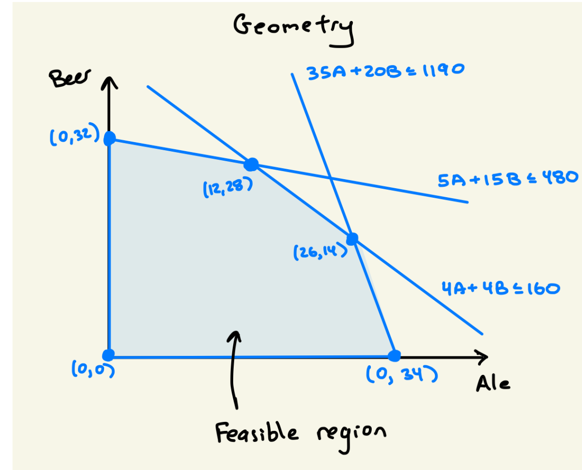
Basic Feasible Solutions:



Basic Feasible Solutions (Algebra)

slack form LP

$$
\begin{aligned}
\max_{A,B,S_C,S_H,S_M} \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B + S_C = 480 \\
& 4A + 4B + S_H = 160 \\
& 35A + 20B + S_M = 1190 \\
& A, B, S_C, S_H, S_M \geq 0
\end{aligned}
$$



constraint matrix

$$
\begin{array}{ccccc}
(A) & (B) & (S_C) & (S_H) & (S_M)
\end{array}
$$

$$
\begin{bmatrix}
5 & 15 & 1 & 0 & 0 \\
4 & 4 & 0 & 1 & 0 \\
35 & 20 & 0 & 0 & 1
\end{bmatrix}
$$

The Simplex Algorithm (30,000' view)

Given an LP in standard form

$$\begin{aligned} \max_{x} \quad & c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Simplex algorithm

- Start with a BFS x_0 corresponding to constraint set S_0

How?

- Repeat until optimality:

How?

  - Find an adjacent BFS x_i corresponding to constraint set S with

$$c^T x_i \geq c^T x_{i-1}$$

Thm: Only terminates at an optimal solution

Simplex in Practice

Theory: Might need exponentially many pivots to terminate

Practice: Can solve LPs with millions of variables/constraints (usually $\leq 2(n+m)$ pivots)

Many Issues to Resolve:

1. What if the LP in infeasible/unbounded?
2. How to choose a good pivot rule?
3. How to avoid cycling?
4. How to maintain sparsity?
5. How to be numerically stable?
6. How to preprocess the LP to be smaller?

Solving Linear Programs: Ellipsoid