CS7800: Advanced Algorithms

Class 9: Linear Programming I

- Basic Concepts
- Simplex Method

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Exam Info

Topics

- Greedy
  - "Greedy Stays Ahead"
  - "Exchange Argument"
  - Minimum spanning tree
- Dynamic programming
  - Weighted interval scheduling
  - Segmented least squares
  - Knapsack
  - Bellman Ford
- Flows
  - Definitions/concepts
  - Ford-Fulkerson alg
  - Max Flow/Min Cut Theorem
  - Faster algorithms

Strong Duality Thm

- 90 minutes
- One 8.5" × 11" sheet of notes

Our Favorite Linear Program

How to maximize the brewery's profits?

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

34 ale , 0 beer ⇒ $442

0 ale , 32 beer ⇒ $736

7.5 ale , 29.5 beer ⇒ $776

12 ale, 28 beer ⇒ $800

decision variable

linear objective function

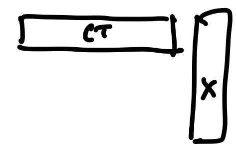
$$\max_{A,B} \; 13A + 23B$$

$$\begin{aligned} 5A + 15B &\le 480 \\ 4A + 4B &\le 160 \\ 35A + 20B &\le 1190 \\ A, B &\ge 0 \end{aligned}$$

linear inequality constraints

Linear Programming

Optimize a linear objective
subject to linear constraints



decision variables

objective

constraints

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^{n} c_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} a_{ij} x_i \le b_j \qquad 1 \le j \le m$$

$$x_i \geq 0 \qquad 1 \le i \le n$$

$$\max_{x \in \mathbb{R}^n} c^T x$$

$$\text{s.t.} \quad Ax \leq b$$

$$x \geq 0$$

$c \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$

always column vectors

Standard Form LPs

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} A_{ji} x_i = b_j \qquad 1 \le j \le m \\
& x_i \geq 0 \qquad 1 \le i \le n
\end{aligned}
$$

Also called <u>slack form</u> for reasons that may be clear soon

Transformation Rules

- Equality to inequality

$$a^T x = b \implies \begin{matrix} a^T x \leq b \\ a^T x \geq b \end{matrix}$$

- Inequality to equality

$$a^T x \leq b \implies \begin{matrix} a^T x + s = b \\ s \geq 0 \end{matrix}$$

- Minimize to maximize

$$\min_x c^T x \implies \max_x -c^T x$$

- Unconstrained to nonnegative

$$x_i \in \mathbb{R} \implies \begin{matrix} x_i = x_i^+ - x_i^- \\ x_i^+, x_i^- \geq 0 \end{matrix}$$

Our Favorite LP (now in standard form)

How to maximize the brewery's profits?

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
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constraint	480	160	1190	

$$\max_{A,B} \quad 13A + 23B$$

$$\text{s.t.} \quad 5A + 15B \le 480$$

$$4A + 4B \le 160$$

$$35A + 20B \le 1190$$

$$A, B \geq 0$$

standard form ("slack form")

$$\max_{A,B,S_C,S_H,S_M} \quad 13A + 23B$$

$$\text{s.t.} \quad 5A + 15B + S_C = 480$$

$$4A + 4B + S_H = 160$$

$$35A + 20B + S_M = 1190$$

$$A, B, S_C, S_H, S_M \geq 0$$

Examples of Linear Programs

Max Flow

$$\max_{\{f(e)\}_{e\in E}} \sum_{e \text{ out of } s} f(e)$$

$$\forall v \quad \sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) = 0 \quad \text{(conservation)}$$

$$f(e) \le c(e) \quad \text{(capacity constraint)}$$

$$f(e) \ge 0 \quad \text{(non-negativity)}$$

Min Cost Flow

$$\min_{\{f(e)\}} \sum_{e\in E} \$(e)\cdot f(e)$$

$$\sum_{e \text{ out of } s} f(e) \ge V \quad \text{(demand)}$$

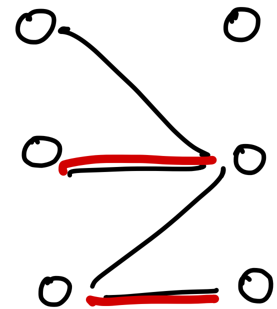
$$\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) = 0 \quad \text{(conservation)}$$

$$f(e) \le c(e) \quad \text{(capacity constraint)}$$

$$f(e) \ge 0 \quad \text{(non-negativity)}$$

Examples of Linear Programs

Max Bipartite Matching



$$\max_{x(e)} \sum_{e \in E} x(e)$$

$$\forall v \quad \sum_{e \text{ incident on } v} x(e) \leq 1$$

$$x(e) \in \{0, 1\}$$

Integer Linear Programming

INTRACTABLE

What can we do for these problems?

① Show that optimal solutions are integral

② Rounding to an integer solution (approximation algorithms)

Geometry of Linear Programs

Algebra

$$\max_{A,B} \quad 13A + 23B$$

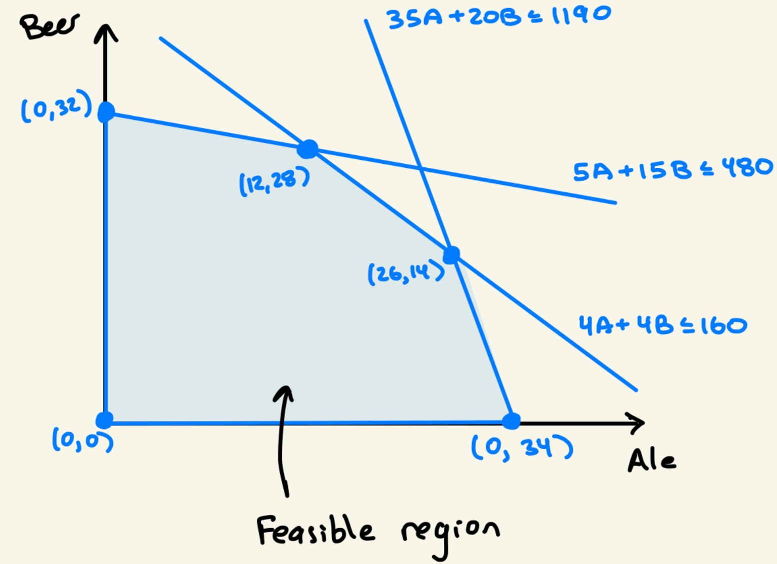
$$\text{s.t.} \quad 5A + 15B \le 480$$

$$4A + 4B \le 160$$

$$35A + 20B \le 1190$$

$$A, B \ge 0$$

Geometry



Geometry of Linear Programs

Algebra

$$\max_{A,B} \quad 13A + 23B$$
$$\text{s.t.} \quad 5A + 15B \le 480$$
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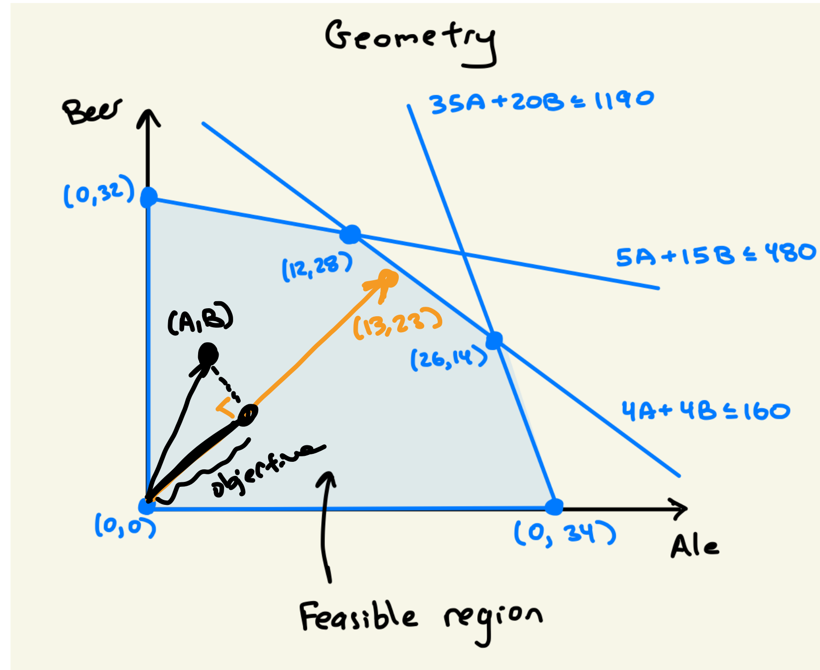
$$A, B \ge 0$$

Goal: Find feasible point with the longest projection onto the line defined by the objective



Geometry of Linear Programs

Algebra

$$\max_{A,B} \quad 13A + 23B$$
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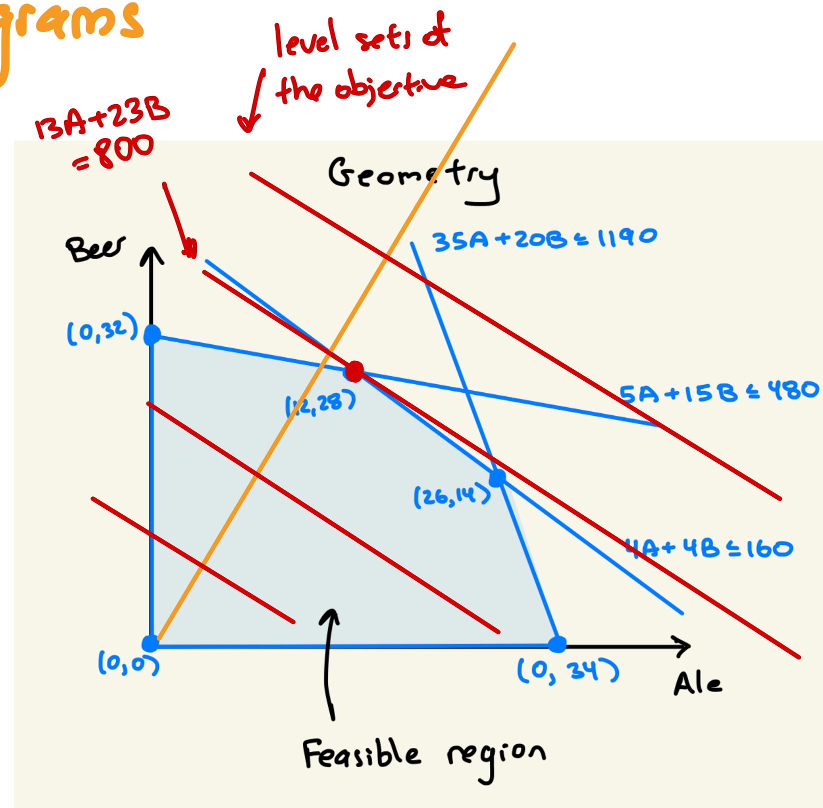


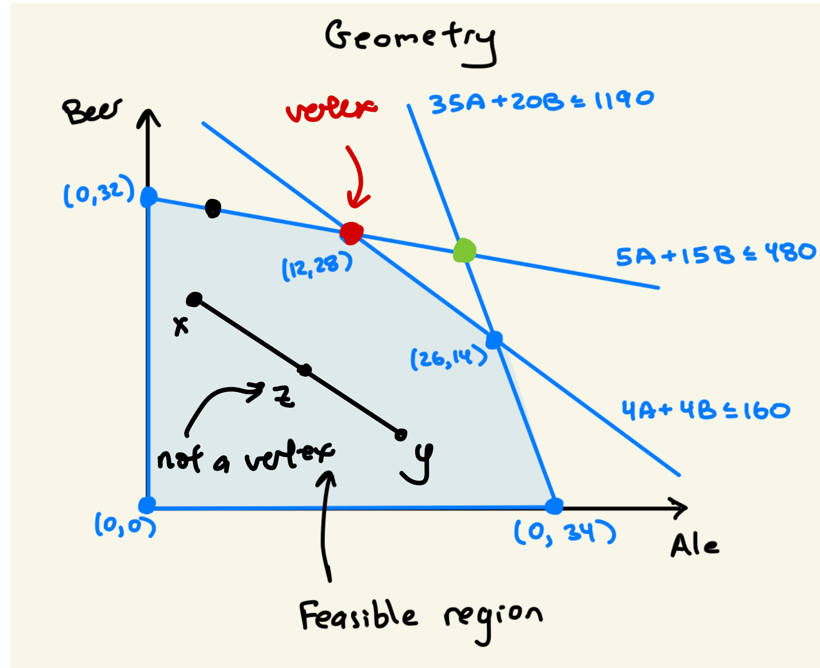
Geometry of Linear Programs

Convexity

A set of points P is <u>convex</u> if for every pair of points $x, y \in P$ and every $\alpha \in [0,1]$

$$\alpha x + (1-\alpha) y \in P$$

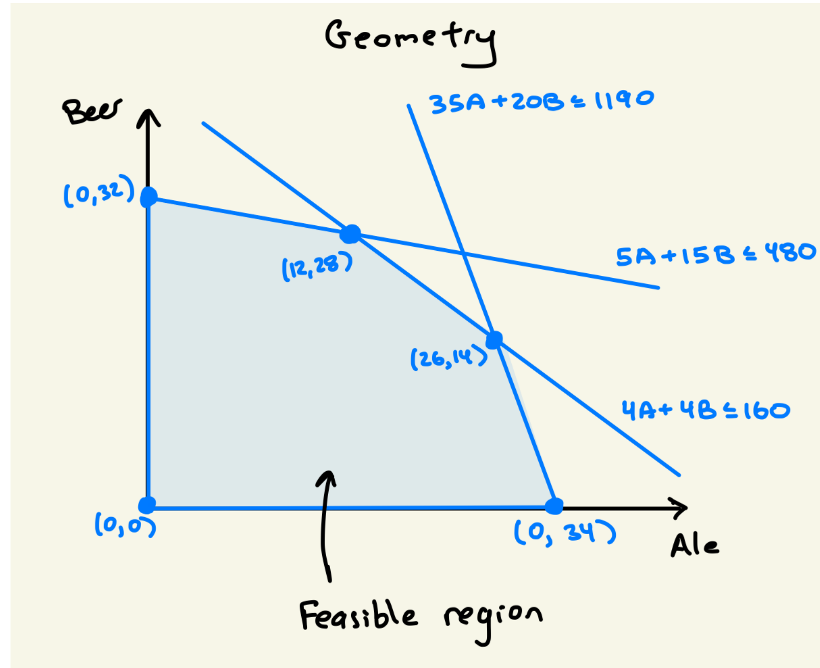
A point $z \in P$ is a <u>vertex of P</u> if z cannot be written as an average of two distinct points $x, y \in P$



<u>Fact</u>: Vertices are formed by the intersection of $\underline{\underline{n}}$ independent constraints.

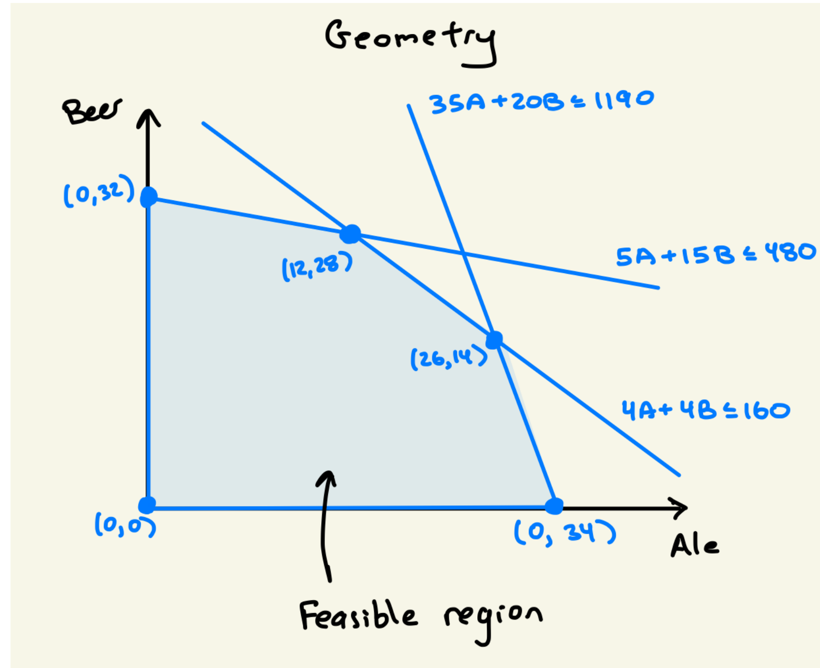
Geometry of Linear Programs

<u>Theorem</u>: If the LP has an optimal solution, then it has an optimal solution at a vertex of the feasible region.



Basic Feasible Solutions (Geometry)

Basic Feasible Solutions:



Basic Feasible Solutions (Algebra)

slack form LP

$$\max_{A,B,S_C,S_H,S_M} \quad 13A + 23B$$

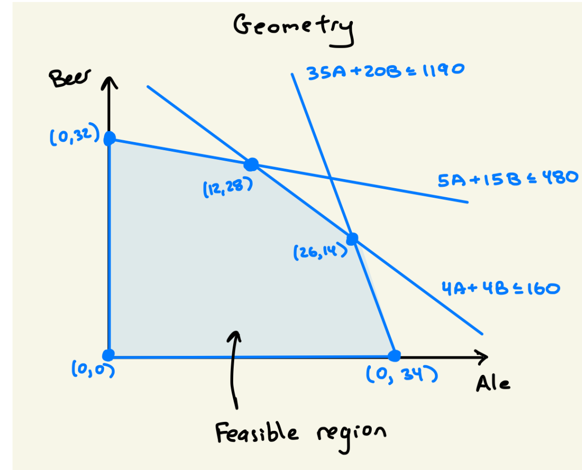
s.t.

$$5A + 15B + S_C = 480$$

$$4A + 4B + S_H = 160$$

$$35A + 20B + S_M = 1190$$

$$A, B, S_C, S_H, S_M \geq 0$$



constraint matrix

$$\begin{array}{ccccc} (A) & (B) & (S_C) & (S_H) & (S_M) \end{array}$$

$$\begin{bmatrix} 5 & 15 & 1 & 0 & 0 \\ 4 & 4 & 0 & 1 & 0 \\ 35 & 20 & 0 & 0 & 1 \end{bmatrix}$$

The Simplex Algorithm (30,000' view)

Given an LP in standard form

$$\max_x \; c^T x$$
$$Ax = b$$
$$x \geq 0$$

Simplex algorithm

- Start with a BFS x_0 corresponding to constraint set S_0

How?

- Repeat until optimality:

How?

  - Find an adjacent BFS x_i corresponding to constraint set S with

$$c^T x_i \geq c^T x_{i-1}$$

Thm: Only terminates at an optimal solution

Simplex in Practice

Theory: Might need exponentially many pivots to terminate

Practice: Can solve LPs with millions of variables/constraints

(usually $\leq 2(n+m)$ pivots)

Many Issues to Resolve:

1. What if the LP in infeasible/unbounded?
2. How to choose a good pivot rule?
3. How to avoid cycling?
4. How to maintain sparsity?
5. How to be numerically stable?
6. How to preprocess the LP to be smaller?