CS 7800: Advanced Algorithms

Class 7: Network Flow I

- Ford-Fulkerson
- Duality

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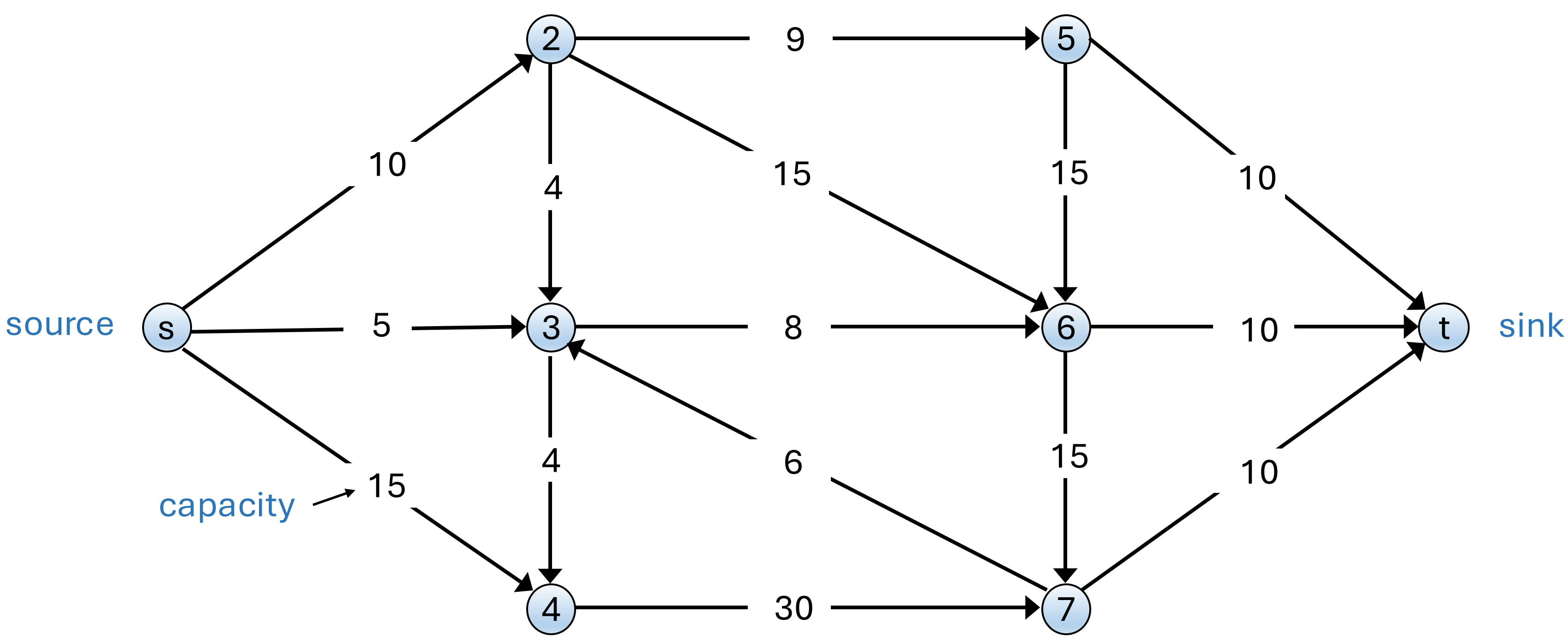
September 26, 2025

Flow Networks

- Directed graph $G = (V, E)$
- Two special nodes: source s and sink t
- Edge capacities $c(e) > 0$
- Assume strongly connected (for simplicity)



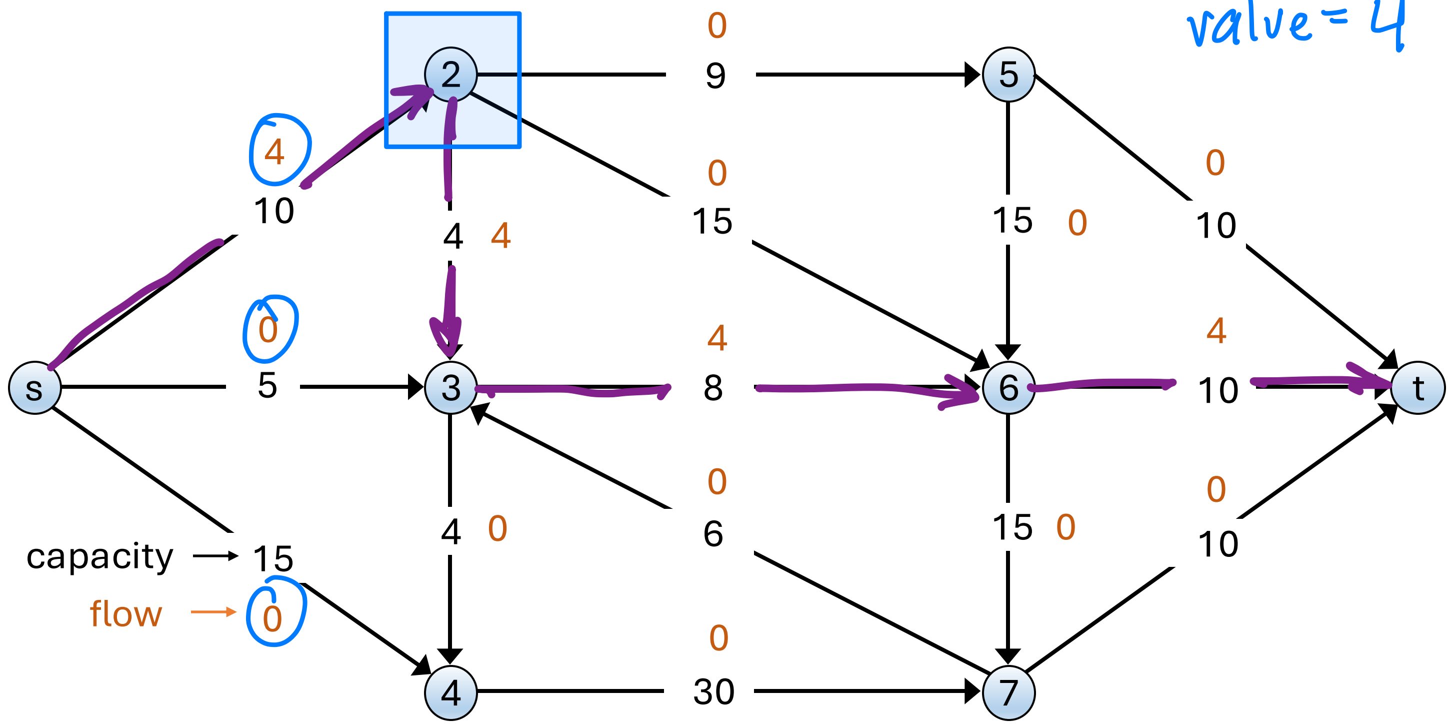
Flows

NB: Sometimes f_e, c_e

- An s-t flow is a function $f(e)$ such that
  - For every $e \in E$, $0 \le f(e) \le c(e)$ (capacity)
  - For every $v \in V \setminus \{s, t\}$, $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

- The value of a flow is $val(f) = \sum_{e \text{ out of } s} f(e)$

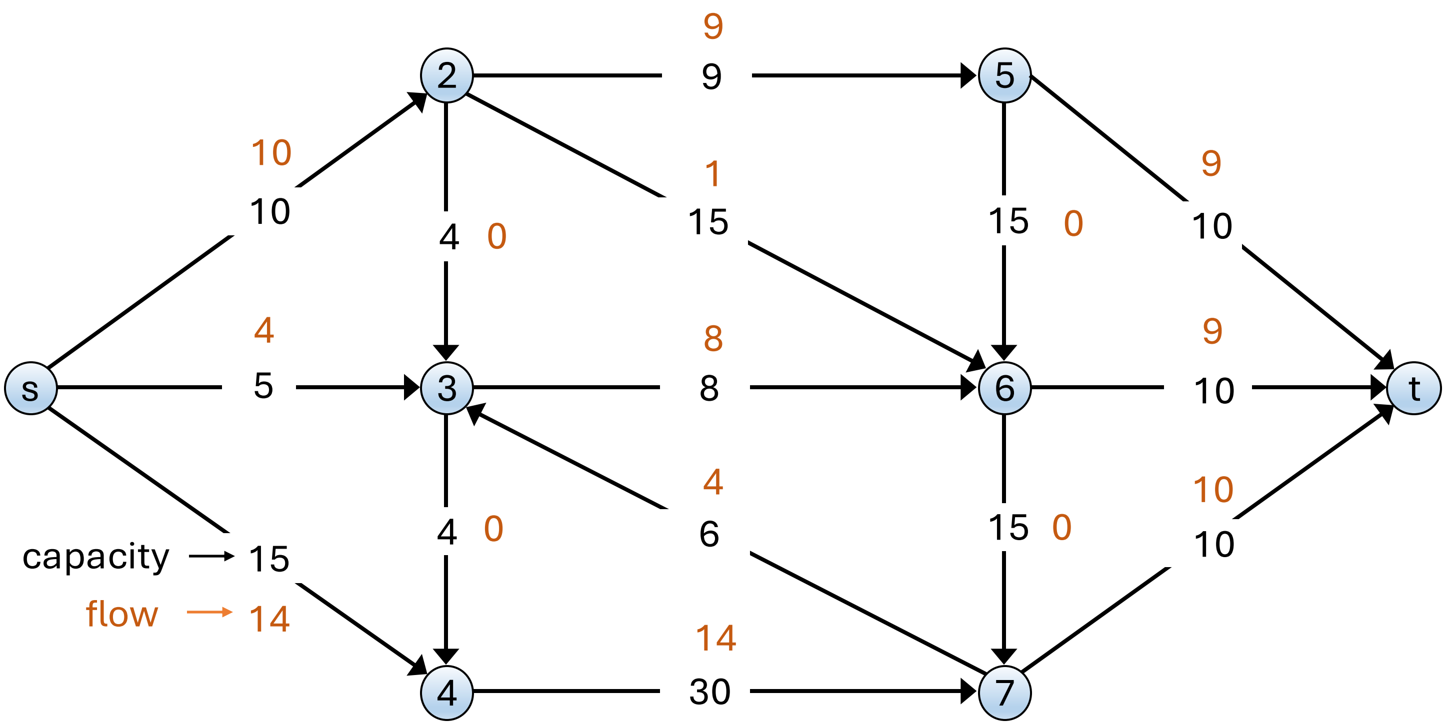
value = 4



Maximum Flow Problem

- Given G = (V,E,s,t,{c(e)}), find an s-t flow of maximum value

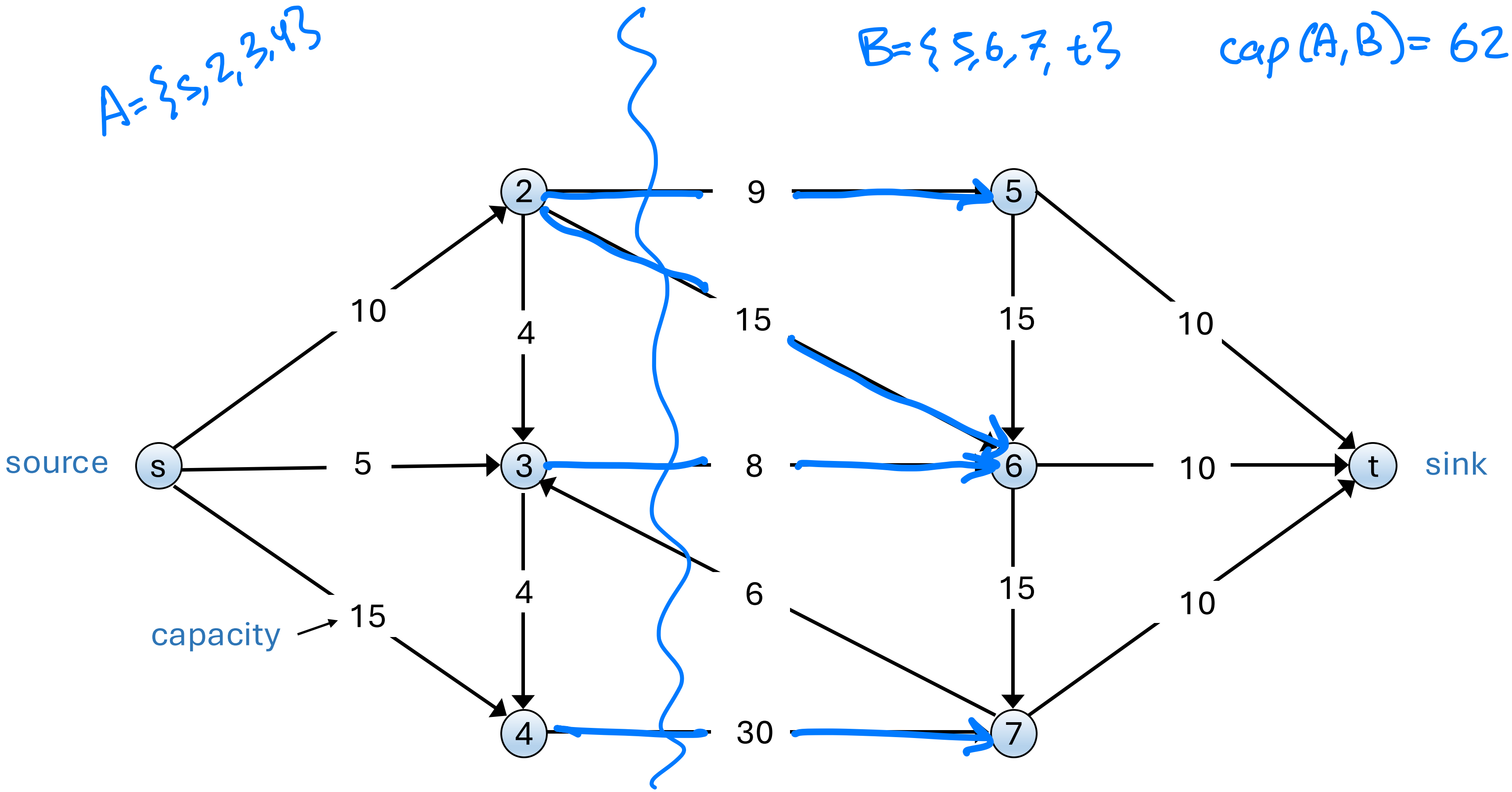
- value(f) = 28



Cuts

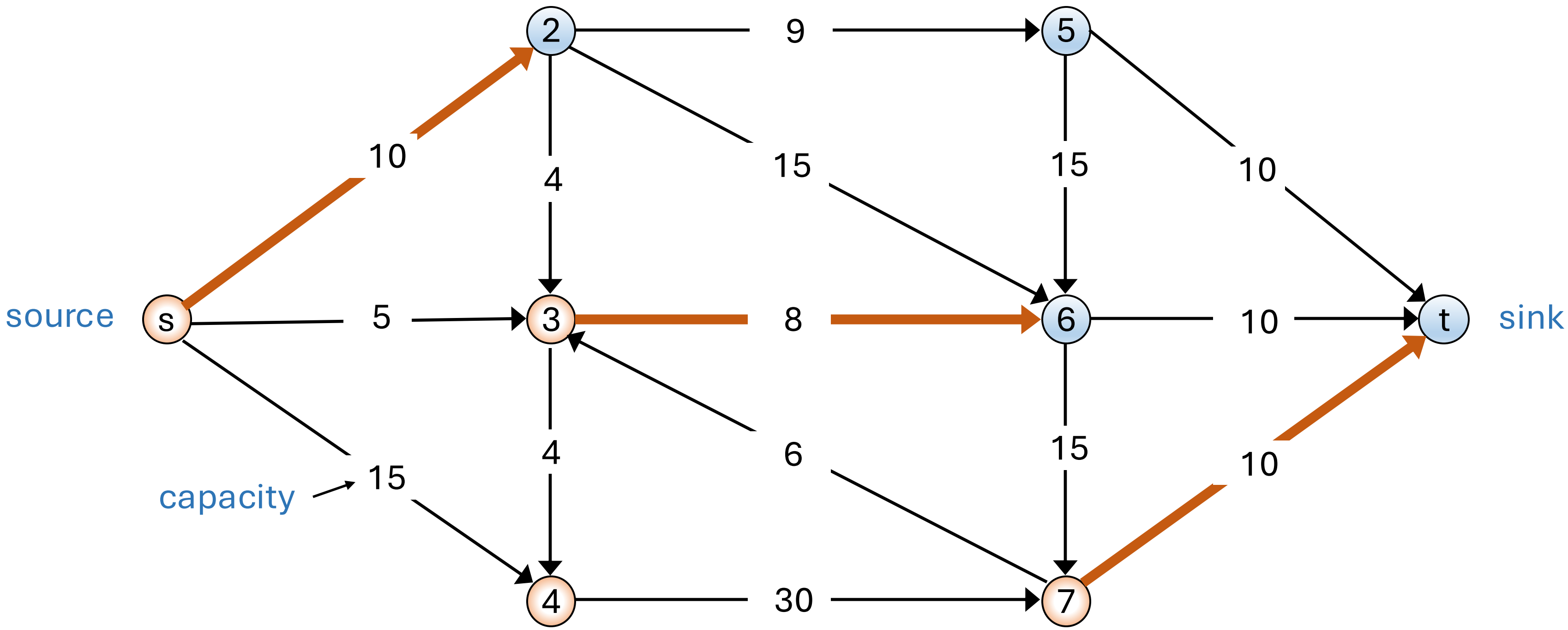
- An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$

- The capacity of a cut (A,B) is $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Minimum Cut problem

- Given G = (V,E,s,t,{c(e)}), find an s-t cut of minimum capacity
- cap({s,3,4,7}, {2,5,6,t}) = 28

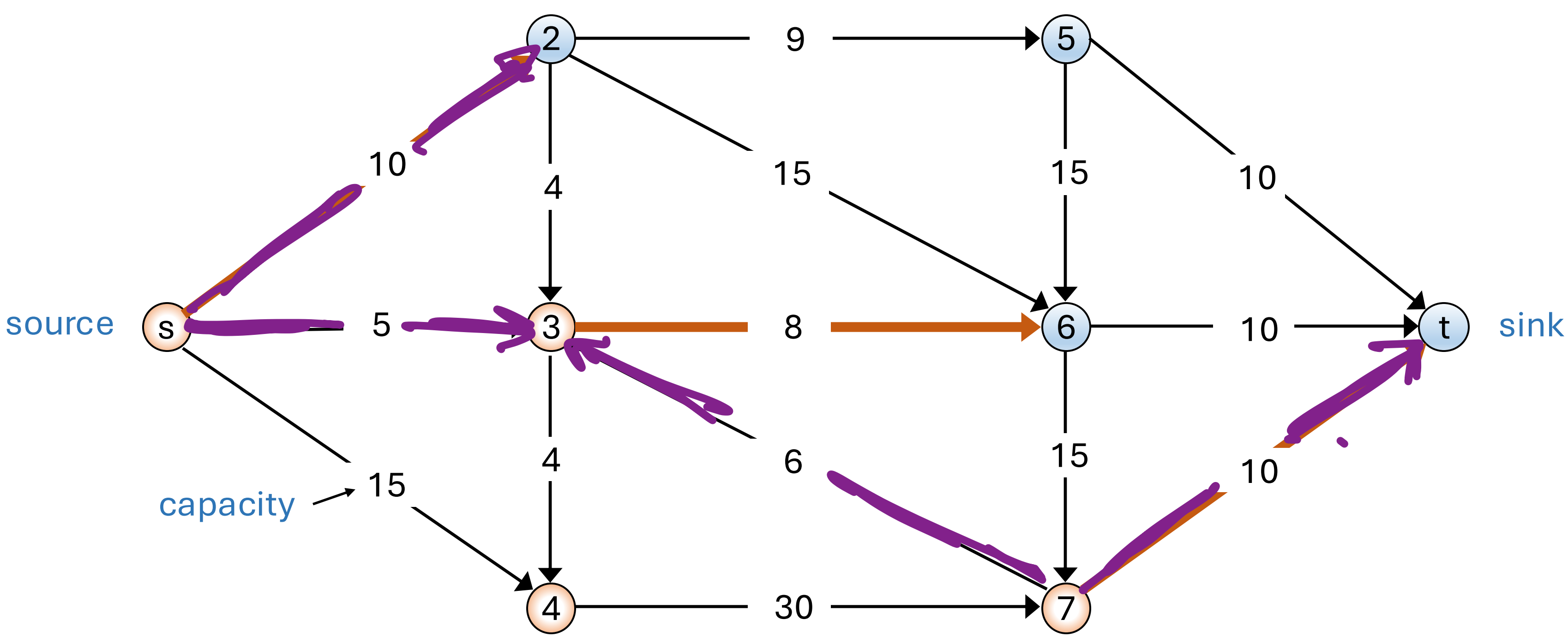


Minimum Cut problem

- Given G = (V,E,s,t,{c(e)}), find an s-t cut of minimum capacity

- cap({s,3,4,7}, {2,5,6,t}) =

A = {s,4,7}   B = {2,3,5,6,t}   cap = 31

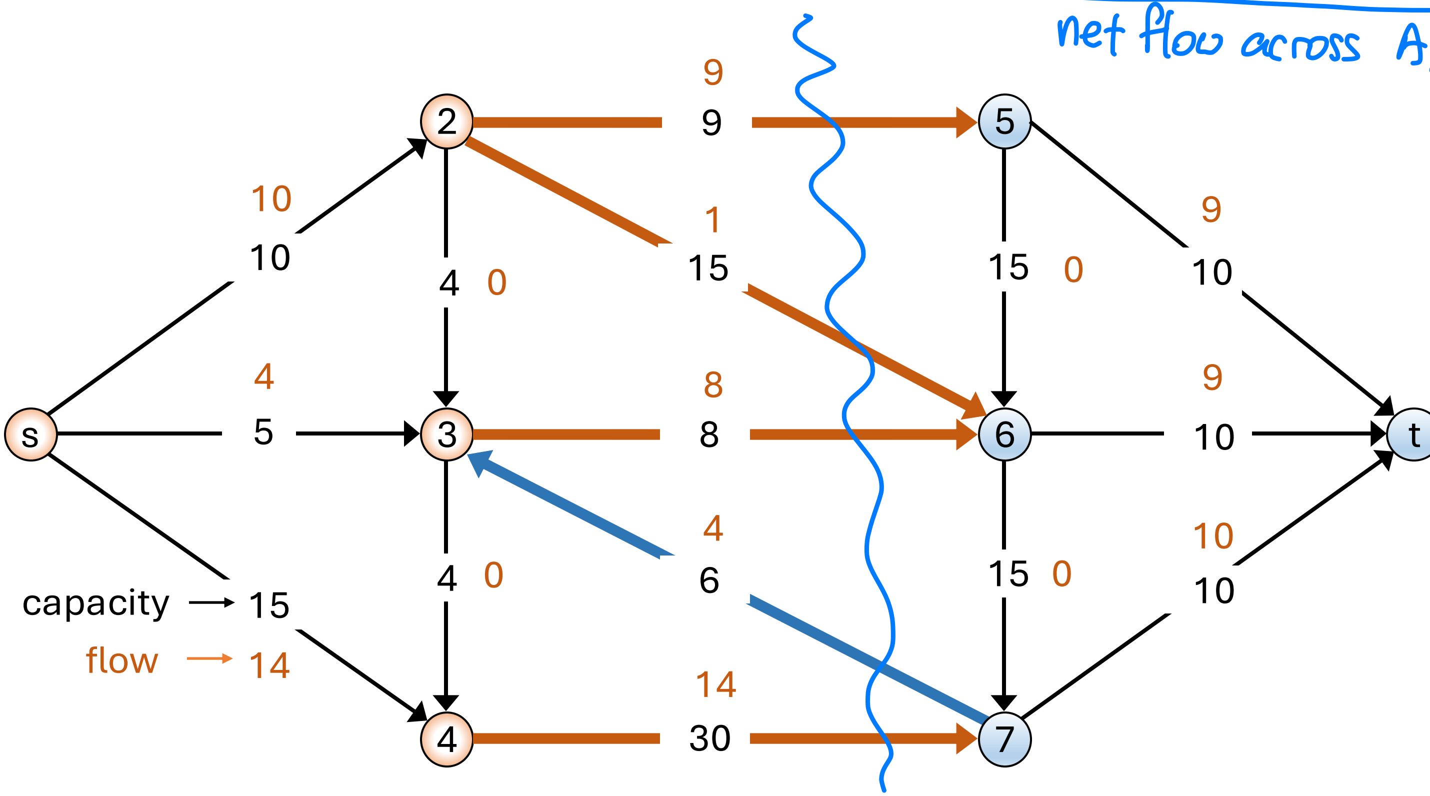


Flows & Cuts: Closely Related

- **Fact:** If f is *any* s-t flow and (A, B) is any s-t cut, then the net flow across (A, B) is equal to the amount leaving s
- The net flow across any s-t cut is the same!

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = val(f)$$

net flow across A,B



Cuts & Flows

Weak Max Flow Min Cut Duality

- Let f be any s-t flow and (A, B) any s-t cut,

$$val(f) \leq cap(A, B)$$

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$ [By fact from prev.]

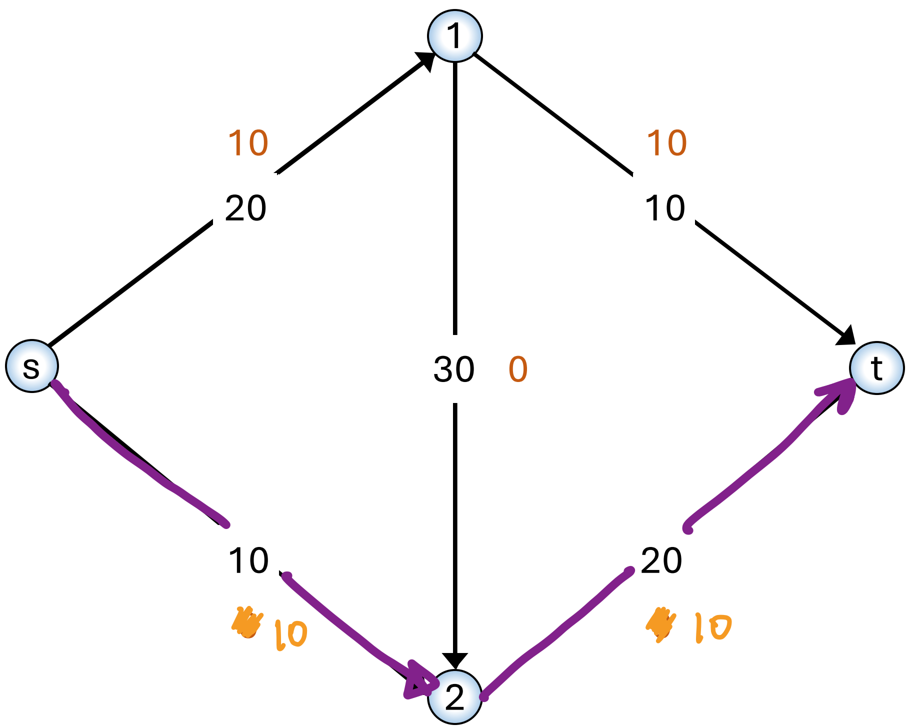
$$\leq \sum_{e \text{ out of } A} f(e)$$ [By non-negativity]

$$\leq \sum_{e \text{ out of } A} c(e)$$ [By capacity]

$$= cap(A, B)$$ [By definition]

Augmenting Paths

- Given a network $G = (V, E, s, t, \{c(e)\})$ and a flow f, an **augmenting path** P is a simple $s \rightarrow t$ path such that $f(e) < c(e)$ for every edge $e \in P$



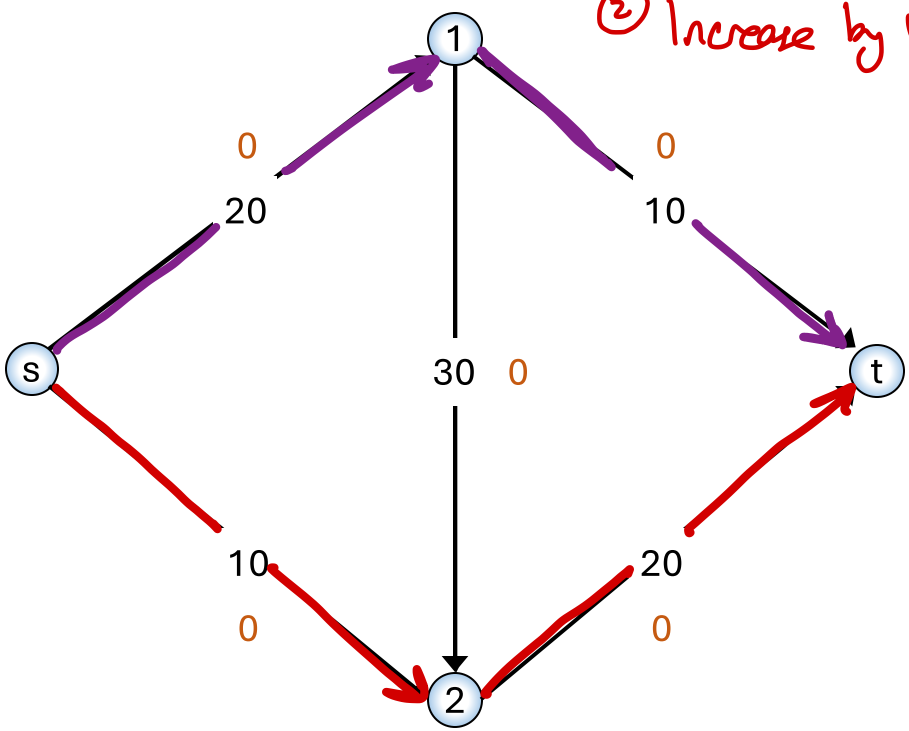
- Are these augmenting paths?
  - s – 1 – t no
  - s – 2 – t yes
  - s – 1 – 2 - t yes

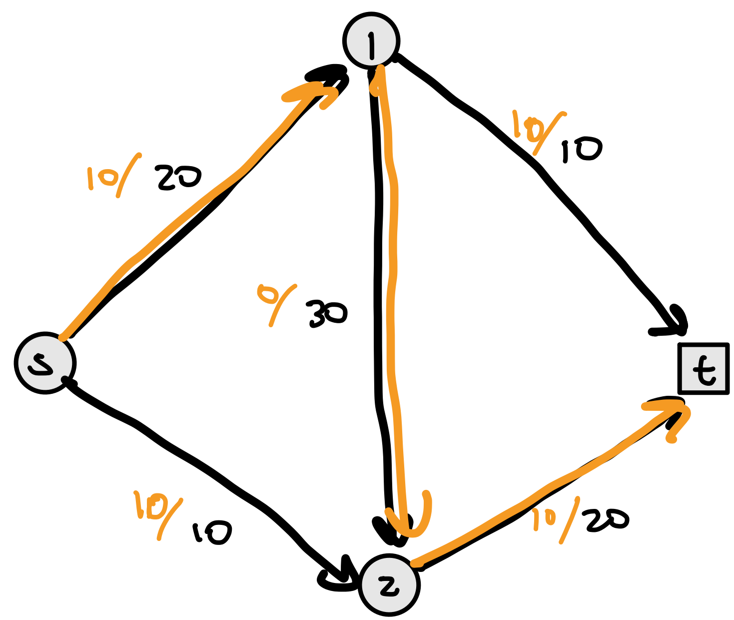
Greedy Max Flow

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P & increase flow
- Repeat until you get stuck

① Increase by 10

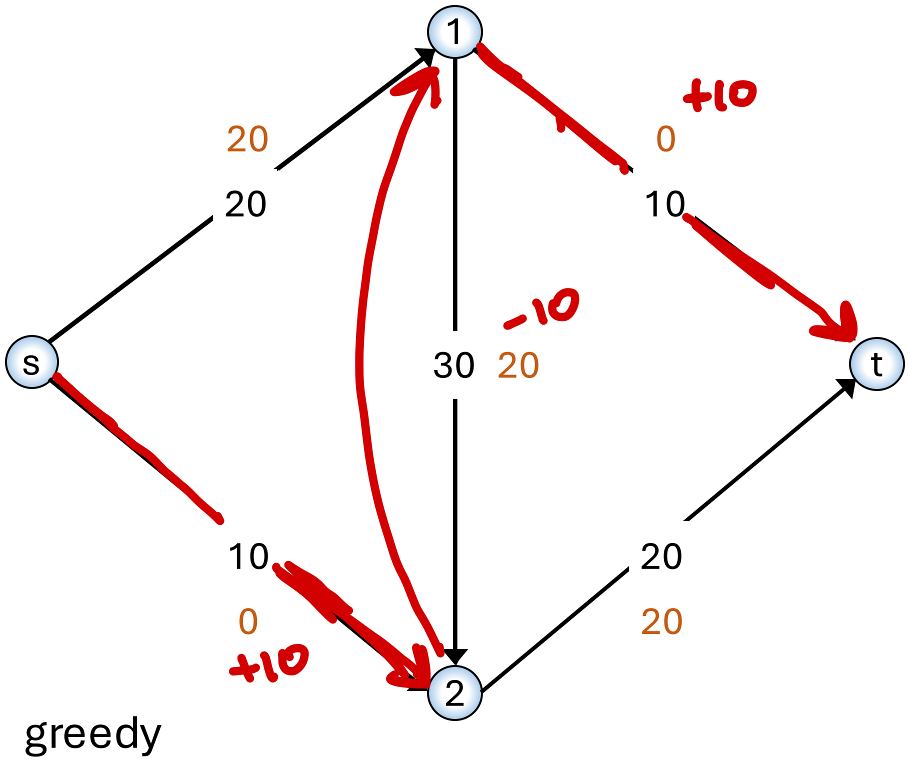
② Increase by 10

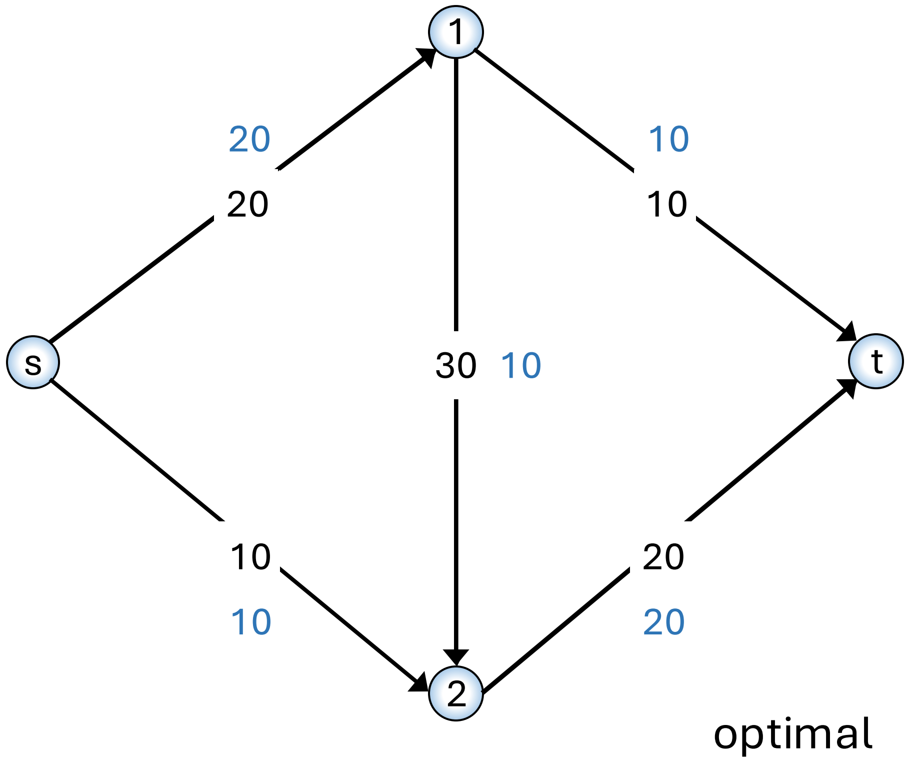




Does Greedy Work?

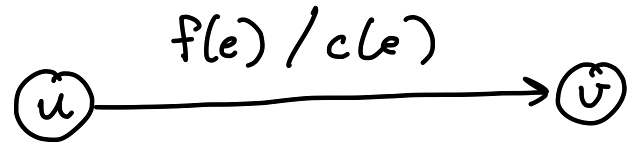
- Greedy gets stuck before finding a max flow
- How can we get from our solution to the max flow?



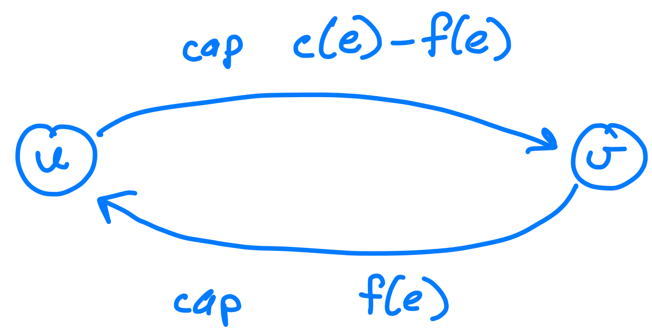


Residual Graphs

- Original edge: $e = (u, v) \in E$.
  - Flow $f(e)$, capacity $c(e)$
  - Residual capacity: $c(e) - f(e)$



- Residual edge
  - Allows "undoing" flow
  - $e = (u, v)$ and $e^R = (v, u)$.
  - cap(e^R) = $f(e)$



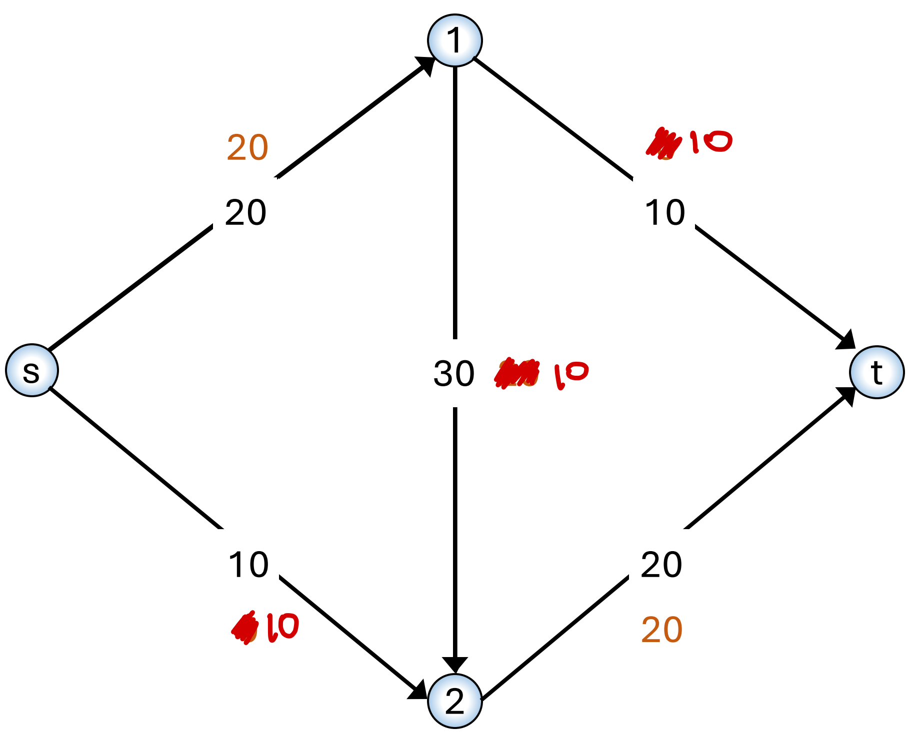
- Residual graph $G_f = (V, E_f)$
  - Original edges with positive residual capacity & residual edges with positive capacity
  - $E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}$.

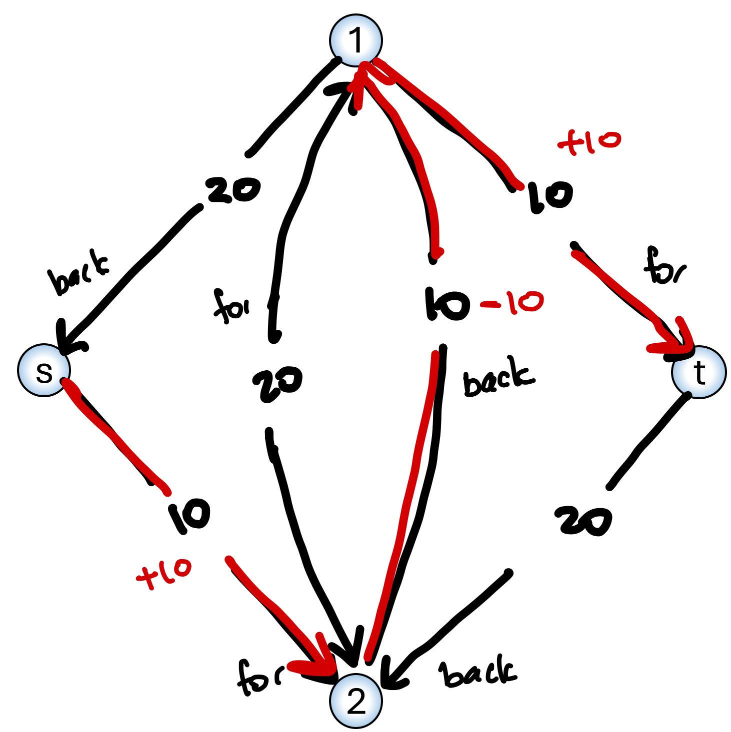
Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Repeat until you get stuck

increase on forward edges

decrease on backward edges





Augmenting Paths in Residual Graphs

- Let G_f be a **residual graph**
- Let P be an augmenting path in the **residual graph**
- **Fact:** $f' = \text{Augment}(G_f, P)$ is a valid flow

```
Augment(G_f, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if (e is an original edge):
            f(e) ← f(e) + b
        else:
            f(e^R) ← f(e^R) - b
    return f
```

- Running Time: $O(n)$

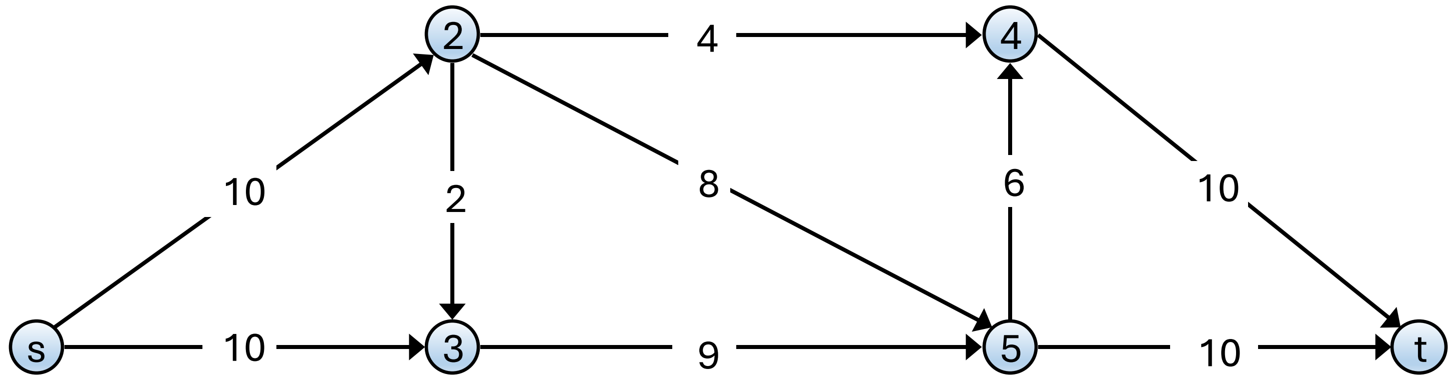
Ford-Fulkerson Algorithm

```
FordFulkerson(G,s,t,{c(e)})
    for e ∈ E: f(e) ← 0                  } // O(m)
    Gf is the residual graph             }

    while (there is an s-t path P in Gf)    }
        f ← Augment(Gf,P) //O(n)    ↖ O(m)  } // O(m) × # of aug paths
        update Gf          //O(n)           }

    return f
```

```
Augment(Gf, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if (e is an original edge): f(e) ← f(e) + b
        else:  f(e^R) ← f( e^R) - b
    return f
```

Ford-Fulkerson Demo

G:



G_f:

What do we want to prove?

- Ford-Fulkerson terminates
  - ↳ After how many augmentations?

- If Ford-Fulkerson terminates then it has max flow

- How can we find a minimum cut?

Running Time of Ford-Fulkerson

- For **integer capacities**, $\leq val(f^*)$ augmentation steps

- Can perform each augmentation step in $O(m)$ time
  - find augmenting path in $O(m)$
  - augment the flow along path in $O(n)$
  - update the residual graph along the path in $O(n)$

- For integer capacities, FF runs in $O\big(m \cdot val(f^*)\big)$ time
  - $O(mn)$ time if all capacities are $c_e = 1$
  - $O(mnC_{\max})$ time for any integer capacities $\leq C_{\max}$
  - Problematic when capacities are large—more on this later!

Optimality of Ford-Fulkerson

- **Theorem:** f is a maximum s-t flow if and only if there is no augmenting s-t path in G_f

- **Strong MaxFlow-MinCut Duality:** The value of the max s-t flow equals the capacity of the min s-t cut

- We'll prove that the following are equivalent for all f
  1. There exists a cut (A, B) such that $val(f) = cap(A, B)$
  2. Flow f is a maximum flow
  3. There is no augmenting path in G_f

Optimality of Ford-Fulkerson

- **Theorem:** the following are equivalent for all f
  1. There exists a cut (A, B) such that $val(f) = cap(A, B)$
  2. Flow f is a maximum flow
  3. There is no augmenting path in G_f

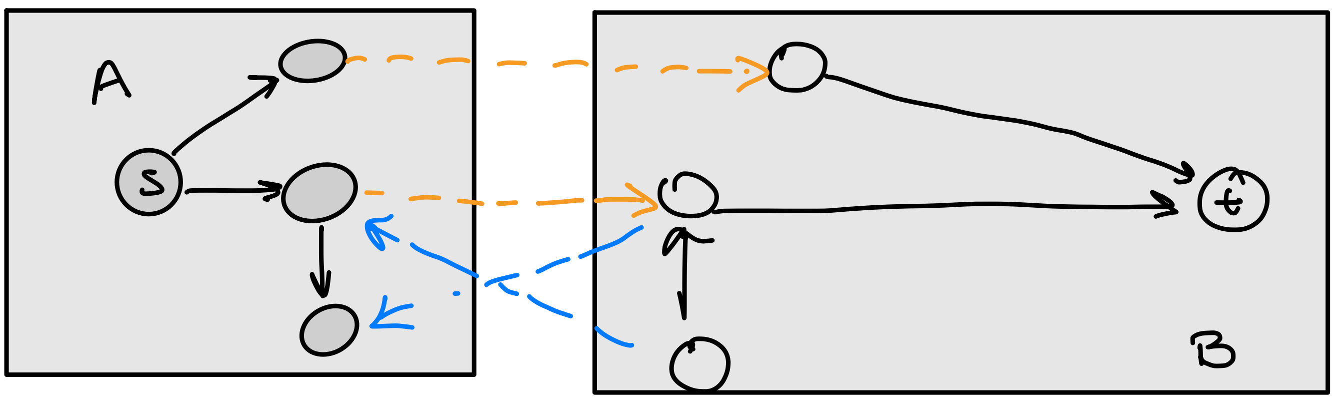
max flow

$1 \Rightarrow 2$: By weak duality $val(f^*) \leq cap(A,B) = val(f)$

$2 \Rightarrow 3$: If there is an augmenting path in G_f then there is a flow $f' = augment(f, P)$ that has higher value

Optimality of Ford-Fulkerson

- **(3 → 1)** If there is no augmenting path in G_f, then there is a cut (A, B) such that $val(f) = cap(A, B)$
  - Let A be the set of nodes reachable from s in G_f
  - Let B be all other nodes



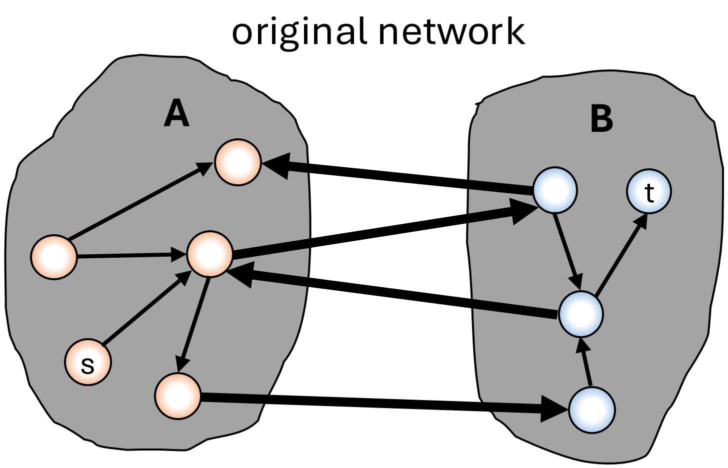
$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ into } A} 0 = cap(A, B)$$

Optimality of Ford-Fulkerson

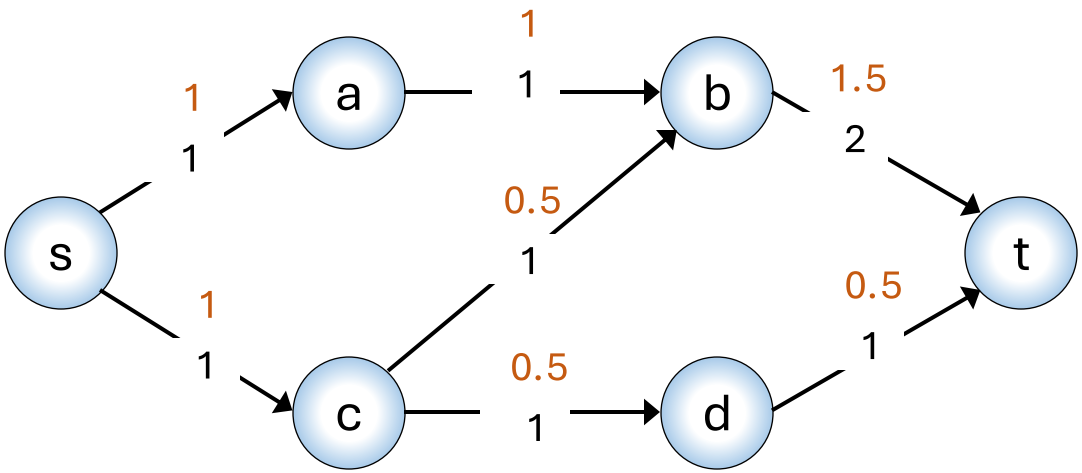
- **(3 → 1)** If there is no augmenting path in G_f, then there is a cut (A, B) such that $val(f) = cap(A, B)$
  - Let A be the set of nodes reachable from s in G_f
  - Let B be all other nodes
  - **Key observation:** no edges in G_f go from A to B

- If e is $A \rightarrow B$, then $f(e) = c(e)$
- If e is $B \rightarrow A$, then $f(e) = 0$



Ask the Audience

- Is this a maximum flow?



- Is there an **integer maximum flow**?
- Does every graph with **integer capacities** have an **integer maximum flow**?

Summary

- **The Ford-Fulkerson Algorithm solves maximum s-t flow**
  - Running time $O\left(m \cdot val(f^*)\right)$ in networks with integer capacities
- **Strong MaxFlow-MinCut Duality: max flow = min cut**
  - The value of the max s-t flow equals the capacity of the min s-t cut
  - If f^* is a maximum s-t flow, then the set of nodes reachable from s in G_{f^*} gives a minimum cut
  - Given a max-flow, can find a min-cut in time $O(n + m)$
- **Every graph with integer capacities has an integer maximum flow**
  - Ford-Fulkerson will return an integer maximum flow
  - Will be super important later