CS 7800: Advanced Algorithms

Class 5: Dynamic Programming II

- Finish Weighted Interval Scheduling
- Segmented Least Squares

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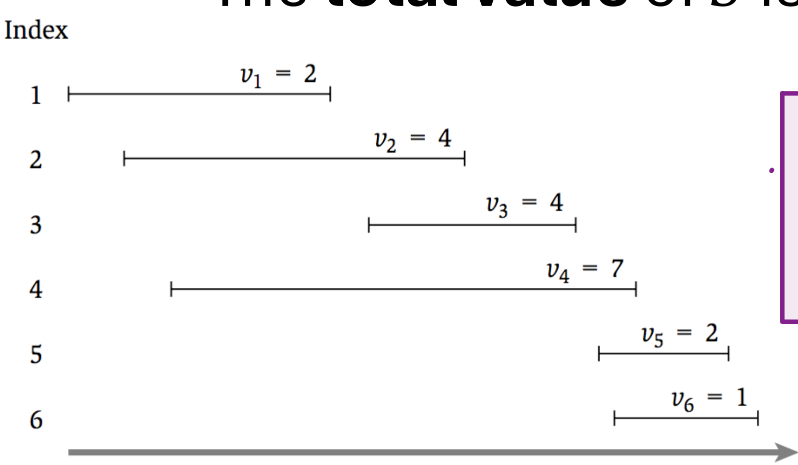
Weighted Interval Scheduling

- **Input:** n intervals (s_i, f_i) each with value v_i
  - Assume intervals are sorted so $f_1 < f_2 < \cdots < f_n$
- **Output:** a compatible schedule S **maximizing** the total value of all intervals
  - A **schedule** is a subset of intervals $S \subseteq \{1, \ldots, n\}$
  - A schedule S is **compatible** if no $i, j \in S$ overlap
  - The **total value** of S is $\sum_{i \in S} v_i$



Notation: p_j = largest i such that $f_i < s_j$

$p_6 = 3$

Finding the Recurrence

"Trick": start by finding only the value of the optimal solution

Subproblems:

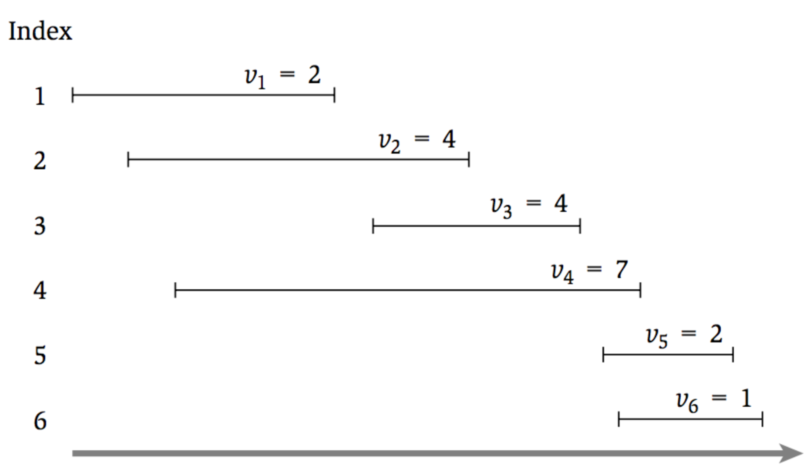
OPT(i) = value of the optimal schedule among intervals 1, 2, ..., i for i = 0, 1, ..., n

Recurrence:

$$OPT(i) = \max\{\underbrace{OPT(i-1)}_{\text{best solution not using } i},\ \underbrace{v_i + OPT(p_i)}_{\text{best solution using } i}\}$$

$$OPT(0) = 0$$

$$OPT(1) = v_1$$



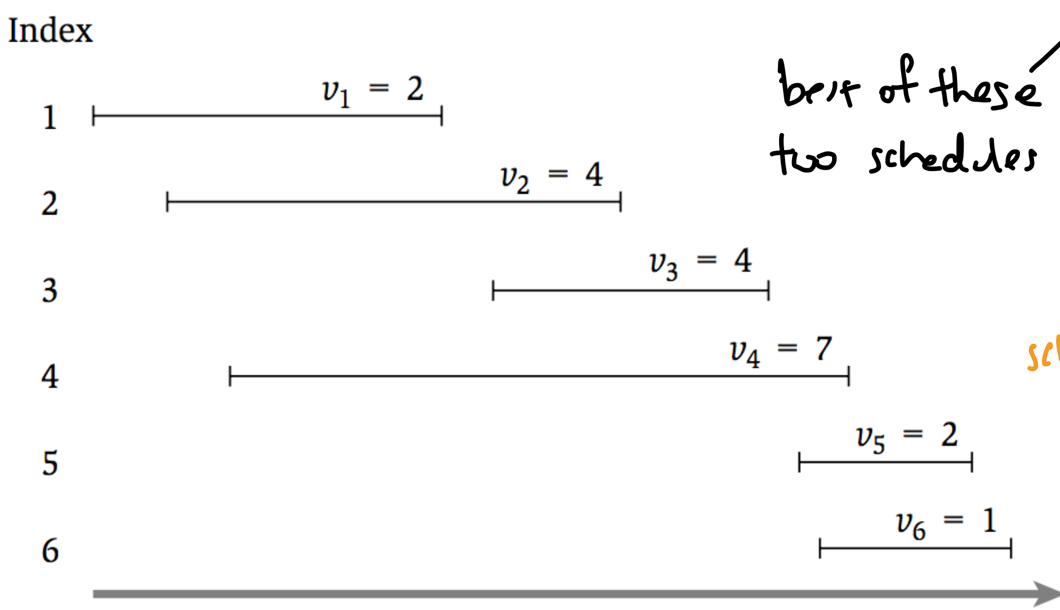
best solution not using i

best solution using i

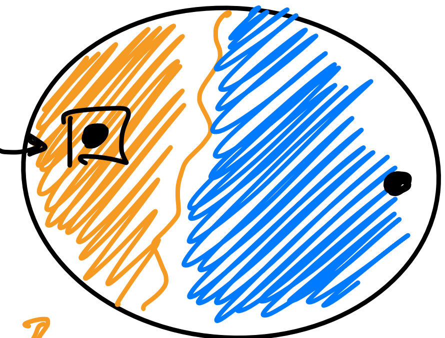
Finding the Optimal Solution

all schedules

But we want a schedule, not a value!



best of these two schedules



schedules w/o i

schedules w i

OPT(0)	OPT(1)	OPT(2)				OPT(6)
M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	M[6]
0	2	4	6	7	8	8

$$OPT(6) = \max\{OPT(5),\ 1 + OPT(3)\}$$

$$= \max\{8,\ 7\}$$

Value of optimal solution w 6

Finding the Solution

Assume that we already filled $M[i] = OPT(i)$

FindSched(n):

- if $n=0$ return $\emptyset$
- elif $n=1$ return $\{1\}$
- else:
  - if $M[n] = M[n-1]$ then return FindSched$(n-1)$
  - if $M[n] = v_n + M[p_n]$ then return $\{n\}$ + FindSched(p_n)

Runtime: $O(n)$

Weighted Interval Scheduling Recap

- There is an $O(n \log n)$ algorithm for the weighted interval scheduling problem
  - Generalizes the greedy alg for the unweighted version
  - Our first example of dynamic programming

- **Dynamic Programming Recipe:**

The "hard" part

(1) identify a set of **subproblems**

(2) relate the subproblems via a **recurrence**

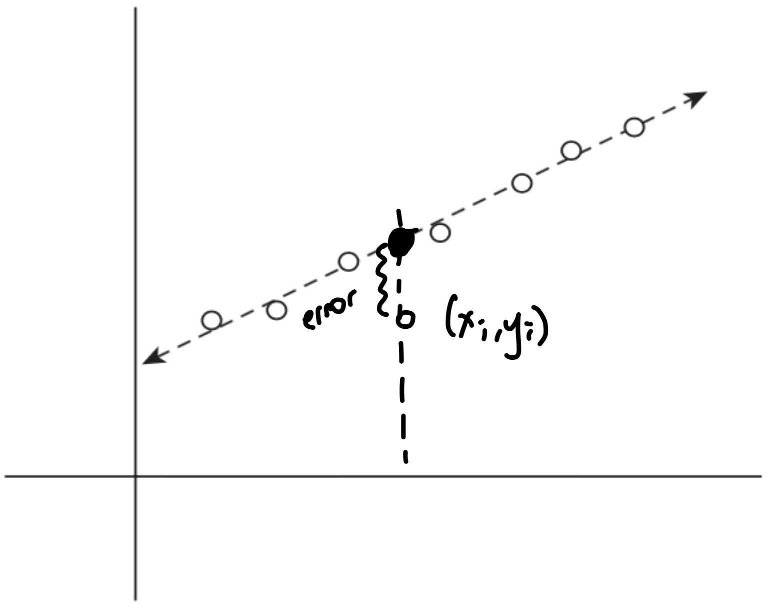
(3) design an algorithm to **efficiently solve** the recurrence

(4) recover the **actual solution** at the end

(Ordinary) Least Squares

$x_1 < x_2 < \dots < x_n$

- **Input:** n data points $P = \{(x_1, y_1), \dots, (x_n, y_n)\}$
- **Output:** the line L (i.e. $y = ax + b$) that fits **best**
  - **best** = minimizes $error(L, P) = \sum_i (y_i - ax_i - b)^2$



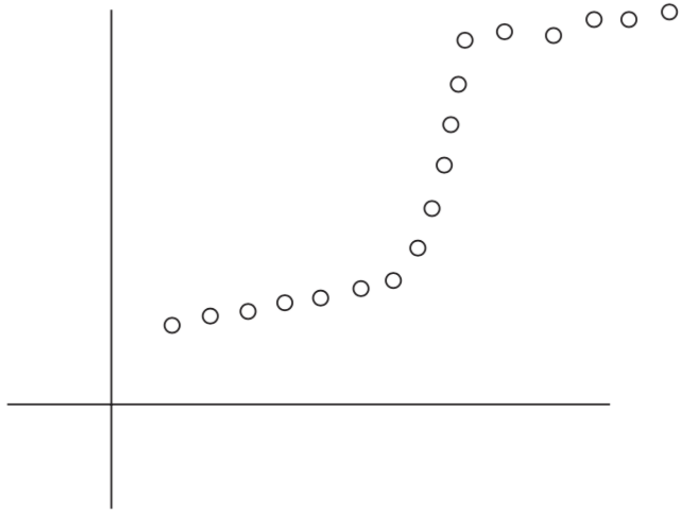
$$a = \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}$$

$$b = \frac{\sum y_i - a\sum x_i}{n}$$

- There is an $O(n)$ time algorithm for finding the line of best fit

Segmented Least Squares

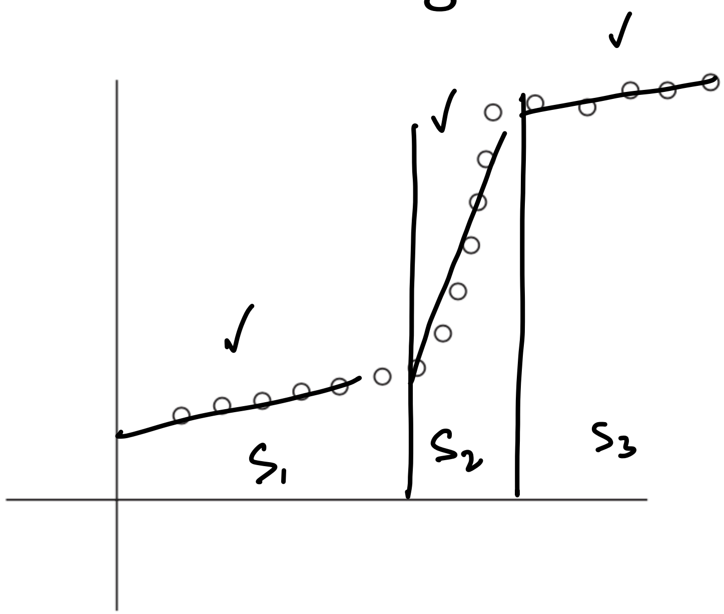
- **Input:** n data points $P = \{(x_1, y_1), \ldots, (x_n, y_n)\}$
- What if the data does not look like a line?



- Some data can be described better by more than one line **segment**
- But, using $\geq {}^{n}/_{2}$ segments defeats the purpose!

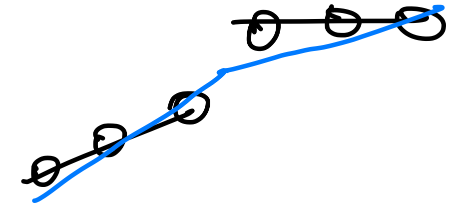
Segmented Least Squares

- **Input:** n data points $P = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, **cost parameter** $C > 0$
  - Assume $x_1 < x_2 < \cdots < x_n$
- **Output:** a partition of P into contiguous (disjoint) segments $S_1, S_2, \ldots, S_m$, lines $L_1, L_2, \ldots, L_m$, minimizing total "cost"

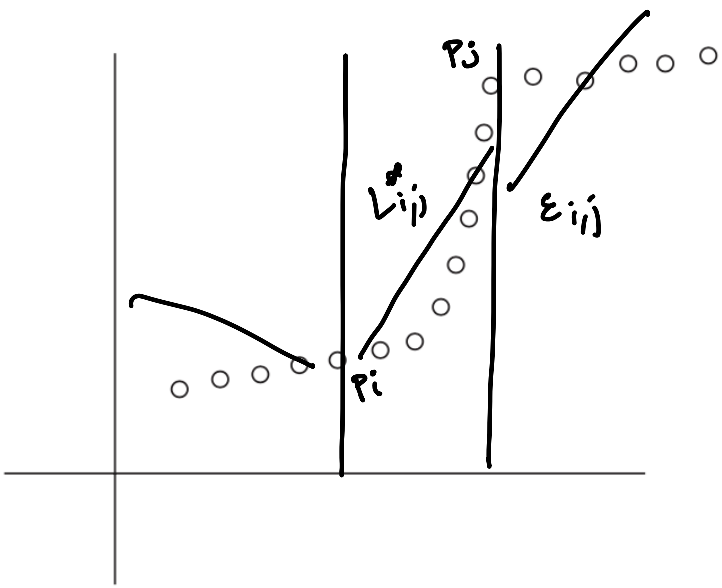


cost($S_1, \ldots, S_m, L_1, \ldots, L_m$)
$= mC + \sum_{i=1}^{m} error(L_i, S_i)$

Segmented Least Squares



- **First observation:** for every segment S_j, L_j must be the (single) line of best fit for S_j
  - Let $L^*_{i,j}$ be the optimal line for $\{p_i, \dots, p_j\}$
  - Let $\varepsilon_{i,j} = error\left(L^*_{i,j}, \{p_i, \dots, p_j\}\right)$



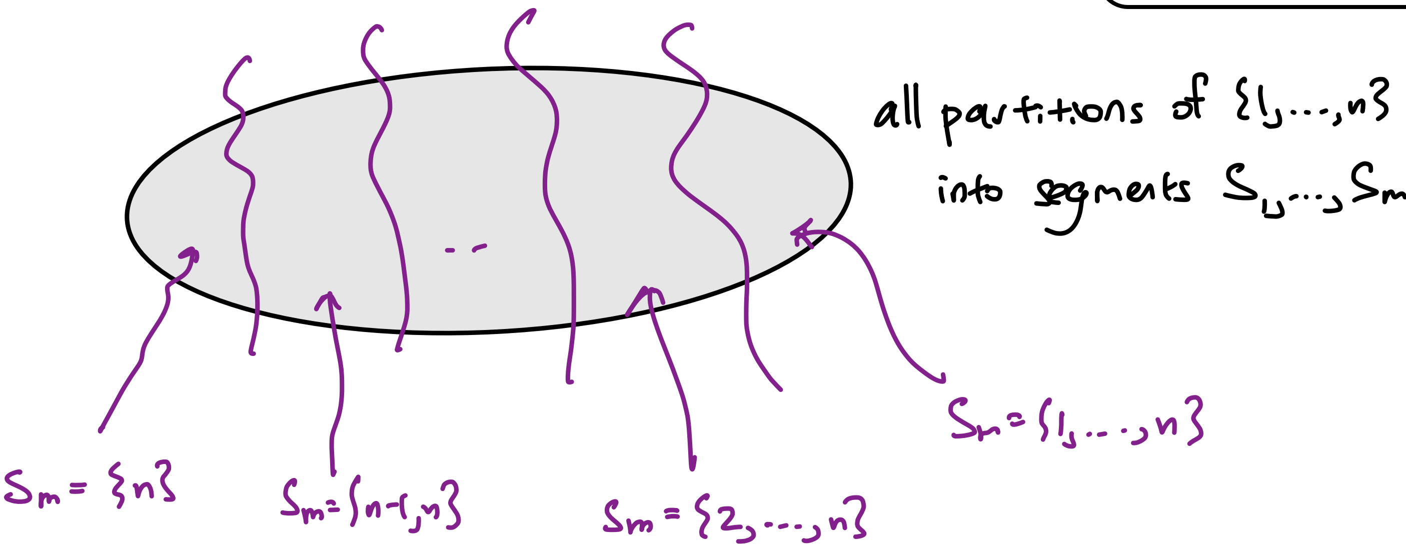
Can compute $\varepsilon_{i,j}$ for all i, j in $O(n^3)$ time straightforwardly,

...or $O(n^2)$ time with more cleverness

Writing the Recurrence

Let $L_{i,j}^*$ be the optimal line for $\{p_i, \dots, p_j\}$

Let $\varepsilon_{i,j} = error(L_{i,j}^*, \{p_i, \dots, p_j\})$

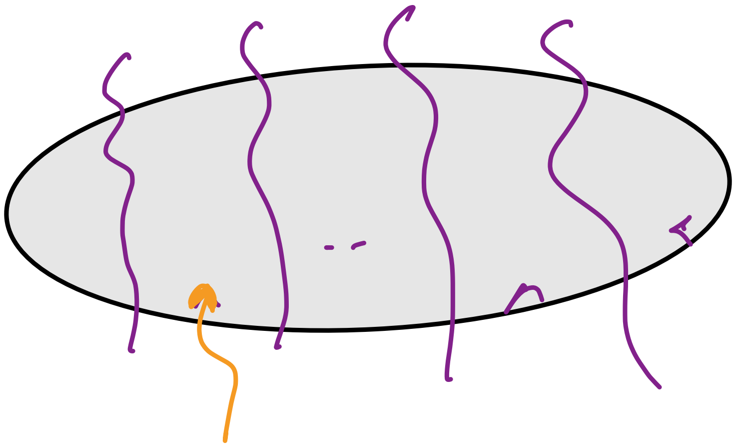


Key Idea: Split possible solutions based on what the last interval is.

Writing the Recurrence

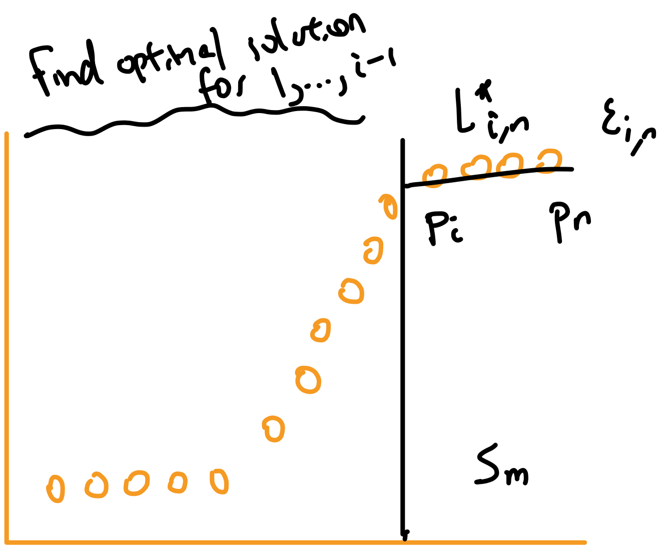
Let $L^*_{i,j}$ be the optimal line for $\{p_i, \ldots, p_j\}$

Let $\varepsilon_{i,j} = error(L^*_{i,j}, \{p_i, \ldots, p_j\})$



all partitions of $\{1, \ldots, n\}$ into segments $S_1, \ldots, S_m$

Type i solutions: Final interval is $\{i, \ldots, n\}$



Best type i solution has cost

$+ \; C$

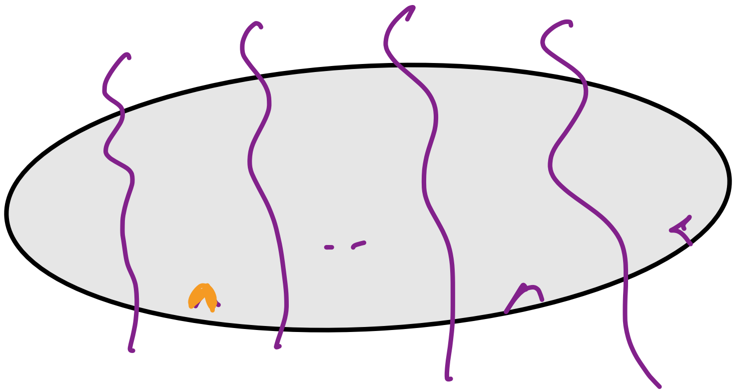
$+ \; \varepsilon_{i,n}$

$+$ cost of optimal SLS for $1, \ldots, i-1$

Writing the Recurrence

Let $L_{i,j}^*$ be the optimal line for $\{p_i, \ldots, p_j\}$

Let $\varepsilon_{i,j} = error(L_{i,j}^*, \{p_i, \ldots, p_j\})$



all partitions of $\{1, \ldots, n\}$ into segments $S_1, \ldots, S_m$

Subproblems: for $i = 0, 1, \ldots, n$ $\quad OPT(i)$ = value of the optimal SLS solution for points $\{1, \ldots, i\}$

Recurrence:

$$OPT(n) = \min_{1 \le i \le n} \{ \underbrace{C + \varepsilon_{i,n} + OPT(i-1)}_{\text{optimal value for solutions of type } i} \}$$

max over each type of solution i

$OPT(0) = 0$

$OPT(1) = C$

$OPT(2) = C$

SLS: Take I

```
// All inputs are global vars
FindOPT(n):
  if (n = 0): return 0
  elseif (n = 1,2): return C
  else:
    return
```
$$\min_{1\le i\le n}\left(\varepsilon_{i,n}+C+\mathbf{FindOPT}(i-1)\right)$$

Runtime:

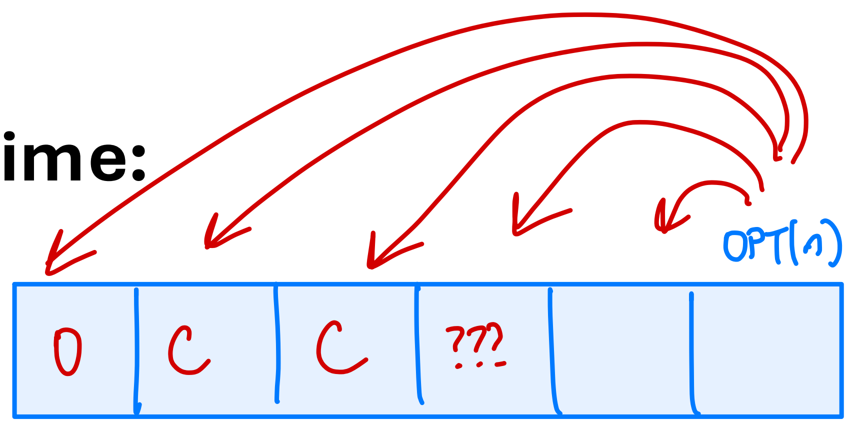
$\Omega(2^n)$

SLS: Take III ("Bottom-Up")

```
// All inputs are global vars
FindOPT(n):
  M[0] ← 0, M[1] ← C, M[2] ← C
  for (j = 3,…,n):
    M[j] ← min_{1≤i≤j}( ε_{i,j} + C + M[i − 1] )
  return M[n]
```

$$\sum_{j=3}^{n} O(j) = O(n^2)$$

Runtime:



Finding Segments

Let $L^*_{i,j}$ be the optimal line for $\{p_i, \dots, p_j\}$

Let $\varepsilon_{i,j} = error(L^*_{i,j}, \{p_i, \dots, p_j\})$

<u>Recurrence:</u>

$$OPT(n) = \min_{1 \le i \le n} \underbrace{\{ C + \varepsilon_{i,n} + OPT(i-1) \}}_{\text{optimal value for solutions of type } i}$$

max over each type of solution i

$OPT(0) = 0$

$OPT(1) = C$

$OPT(2) = C$

- There is some i such that $OPT(n) = C + \varepsilon_{i,n} + OPT(i-1)$
  - "Optimal solution for points $\{1, \dots, n\}$ is the optimal solution for points $\{1, \dots, i-1\}$ combined with final segment $\{i, \dots, n\}$"

Finding Segments

M[i] = OPT(i)

```
// All inputs are global vars
// M[0:n] contains solutions to subproblems
FindSol(M,n):
  if (n = 0): return ∅
  elseif (n = 1): return {1}
  elseif (n = 2): return {1,2}
  else:
```

Let $\cancel{x}$ i $\leftarrow \operatorname{argmin}_{1 \le i \le n} \left(\varepsilon_{i,n} + C + M[i-1]\right)$: which type i gave smallest cost

return {$\cancel{x}$ i, …, n} + FindSol(M, $\cancel{x-1}$ $i-1$)

Runtime: $O(n^2)$

Weighted Interval Scheduling Recap

SLS problem

- There is an $O(n^2)$ time algorithm for the ~~weighted interval scheduling problem~~
  - Second example of dynamic programming
  - Canonical example of partitioning a line into segments

- **Dynamic Programming Recipe:**

(1) identify a set of **subproblems**

(2) relate the subproblems via a **recurrence**

(3) design an algorithm to **efficiently solve** the recurrence

(4) recover the **actual solution** at the end