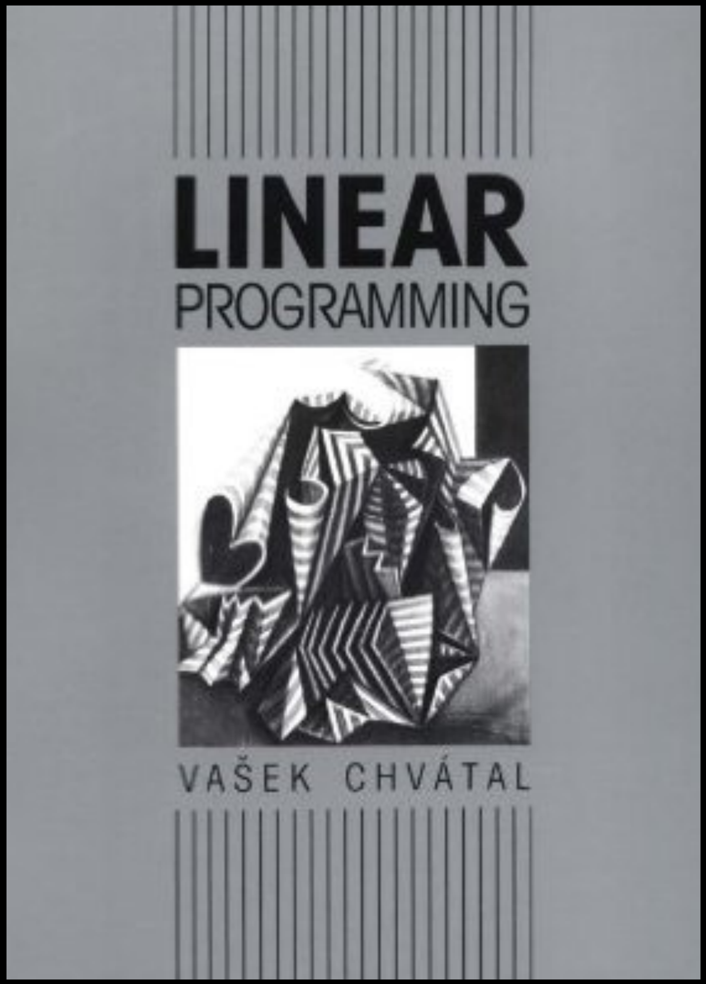
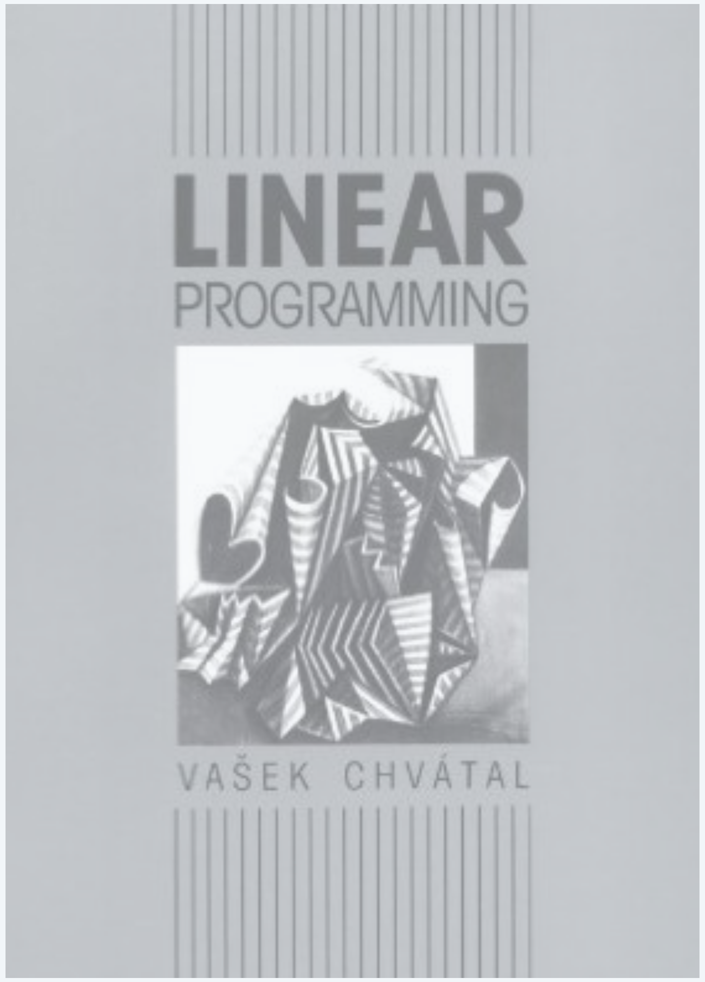


Lecture slides by Kevin Wayne

Linear Programming II

- *LP duality*
- *strong duality theorem*
- *bonus proof of LP duality*
- *applications*

Last updated on 7/25/17 11:09 AM



Linear Programming II

- *LP duality*
- *Strong duality theorem*
- *Bonus proof of LP duality*
- *Applications*

LP duality

Primal problem.

$$\begin{array}{llcccccc} (P) & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \leq & 480 \\ & & 4A & + & 4B & \leq & 160 \\ & & 35A & + & 20B & \leq & 1190 \\ & & A & , & B & \geq & 0 \end{array}$$

Goal. Find a lower bound on optimal value.

Easy. Any feasible solution provides one.

Ex 1. $(A, B) = (34, 0) \Rightarrow z^* \geq 442$

Ex 2. $(A, B) = (0, 32) \Rightarrow z^* \geq 736$

Ex 3. $(A, B) = (7.5, 29.5) \Rightarrow z^* \geq 776$

Ex 4. $(A, B) = (12, 28) \Rightarrow z^* \geq 800$

LP duality

Primal problem.

$$
\begin{array}{llrcrcr}
(P) & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & & 4A & + & 4B & \leq & 160 \\
 & & 35A & + & 20B & \leq & 1190 \\
 & & A & , & B & \geq & 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 1. Multiply 2nd inequality by 6: $24\,A + 24\,B \leq 960.$

$$\Rightarrow \quad z^* = \underbrace{13\,A + 23\,B}_{\text{objective function}} \leq 24\,A + 24\,B \leq 960.$$

LP duality

Primal problem.

$$
\begin{array}{llrcrcr}
(P) & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & & 4A & + & 4B & \leq & 160 \\
 & & 35A & + & 20B & \leq & 1190 \\
 & & A & , & B & \geq & 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 2 times 1st inequality to 2nd inequality: ≤

$$\Rightarrow \quad z^* = 13A + 23B \leq 14A + 34B \leq 1120.$$

LP duality

Primal problem.

$$
\begin{array}{llcccccc}
(P) & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \le & 480 \\
 & & 4A & + & 4B & \le & 160 \\
 & & 35A & + & 20B & \le & 1190 \\
 & & A & , & B & \ge & 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 1 times 1st inequality to 2 times 2nd inequality: ≤

$$\Rightarrow \quad z^* = 13\,A + 23\,B \;\le\; 13\,A + 23\,B \;\le\; 800.$$

Recall lower bound. $(A, B) = (34, 0) \Rightarrow z^* \ge 442$

Combine upper and lower bounds: z* = 800.

LP duality

Primal problem.

$$\begin{array}{llrcrcr}
(P) & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & & 4A & + & 4B & \leq & 160 \\
 & & 35A & + & 20B & \leq & 1190 \\
 & & A & , & B & \geq & 0
\end{array}$$

Idea. Add nonnegative combination (C, H, M) of the constraints s.t.

$$\begin{array}{rcl}
13A + 23B & \leq & (5C + 4H + 35M)\ A \ + \ (15C + 4H + 20M)\ B \\
 & \leq & 480C + 160H + 1190M
\end{array}$$

Dual problem. Find best such upper bound.

$$\begin{array}{llrcrcrcr}
(D) & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s. t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
 & & 15C & + & 4H & + & 20M & \geq & 23 \\
 & & C & , & H & , & M & \geq & 0
\end{array}$$

LP duality: economic interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

$$
\begin{array}{lllllll}
\text{(P)} & \max & 13A & + & 23B & & \\
& \text{s.t.} & 5A & + & 15B & \leq & 480 \\
& & 4A & + & 4B & \leq & 160 \\
& & 35A & + & 20B & \leq & 1190 \\
& & A & , & B & \geq & 0
\end{array}
$$

Entrepreneur: buy individual resources from brewer at min cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if $5C + 4H + 35M < 13$.

$$
\begin{array}{lllllllll}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
& \text{s.t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
& & 15C & + & 4H & + & 20M & \geq & 23 \\
& & C & , & H & , & M & \geq & 0
\end{array}
$$

LP duals

Canonical form.

$$
\text{(P)} \quad \max\ c^T x \quad \text{s.t.} \quad Ax \le b, \quad x \ge 0
$$

$$
\text{(D)} \quad \min\ y^T b \quad \text{s.t.} \quad A^T y \ge c, \quad y \ge 0
$$

Double dual

Canonical form.

$$\text{(P)} \quad \max c^T x \quad \text{s.t.} \quad Ax \le b, \quad x \ge 0$$

$$\text{(D)} \quad \min y^T b \quad \text{s.t.} \quad A^T y \ge c, \quad y \ge 0$$

Property. The dual of the dual is the primal.

Pf. Rewrite (D) as a maximization problem in canonical form; take dual.

$$\text{(D')} \quad \max \ -y^T b \quad \text{s.t.} \quad -A^T y \le -c, \quad y \ge 0$$

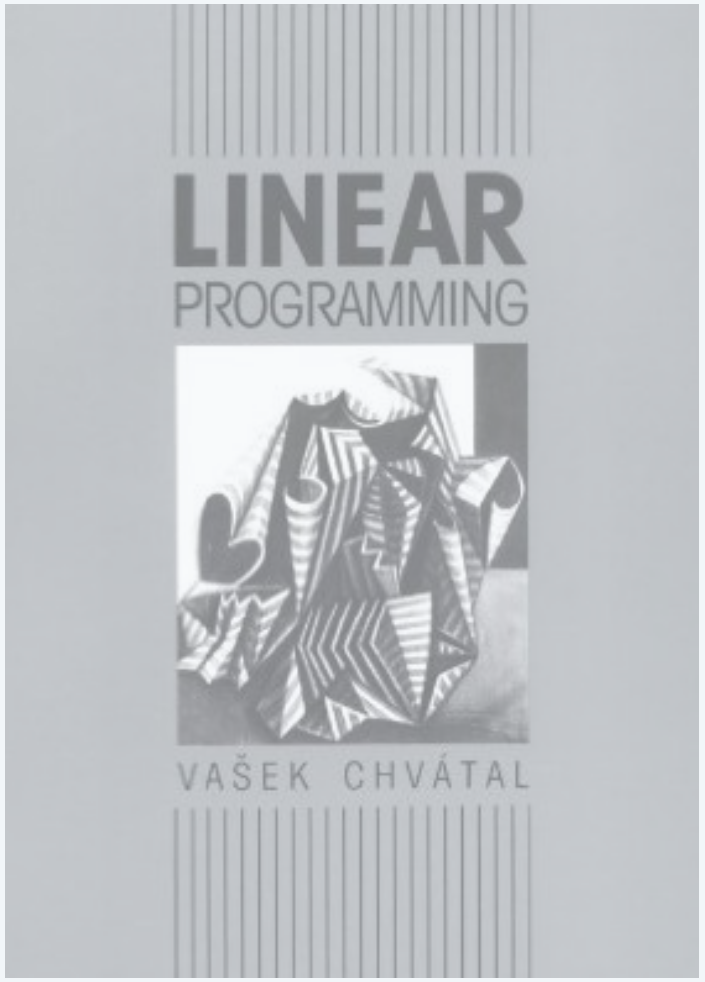
$$\text{(DD)} \quad \min \ -c^T z \quad \text{s.t.} \quad -(A^T)^T z \ge -b, \quad z \ge 0$$

Taking duals

LP dual recipe.

Primal (P)	maximize	minimize	Dual (D)
constraints	$a\,x = b_i$ $a\,x \le b$ $a\,x \ge b_i$	y_i unrestricted $y_i \ge 0$ $y_i \le 0$	variables
variables	$x_j \ge 0$ $x_j \le 0$ unrestricted	$a^{\mathrm{T}}y \ge c_j$ $a^{\mathrm{T}}y \le c_j$ $a^{\mathrm{T}}y = c_j$	constraints

Pf. Rewrite LP in standard form and take dual.



LINEAR PROGRAMMING II

- *LP duality*
- *strong duality theorem*
- *bonus proof of LP duality*
- *applications*

LP strong duality

Theorem. [Gale–Kuhn–Tucker 1951, Dantzig–von Neumann 1947]
For $A \in \Re^{m \times n}, b \in \Re^m, c \in \Re^n$, if (P) and (D) are nonempty, then max = min.

$$\begin{array}{llrcl} \text{(P)} & \max c^T x & & & \\ & \text{s. t.} & Ax & \leq & b \\ & & x & \geq & 0 \end{array}$$

$$\begin{array}{llrcl} \text{(D)} & \min y^T b & & & \\ & \text{s. t.} & A^T y & \geq & c \\ & & y & \geq & 0 \end{array}$$

Generalizes:

- Dilworth's theorem.
- König–Egervary theorem.
- Max-flow min-cut theorem.
- von Neumann's minimax theorem.
- ...

Pf. [ahead]

LP weak duality

Theorem. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty, then max ≤ min.

$$
\begin{array}{llrcl}
\text{(P)} & \max c^T x & & & \\
 & \text{s.t.} & Ax & \le & b \\
 & & x & \ge & 0
\end{array}
\qquad
\begin{array}{llrcl}
\text{(D)} & \min y^T b & & & \\
 & \text{s.t.} & A^T y & \ge & c \\
 & & y & \ge & 0
\end{array}
$$

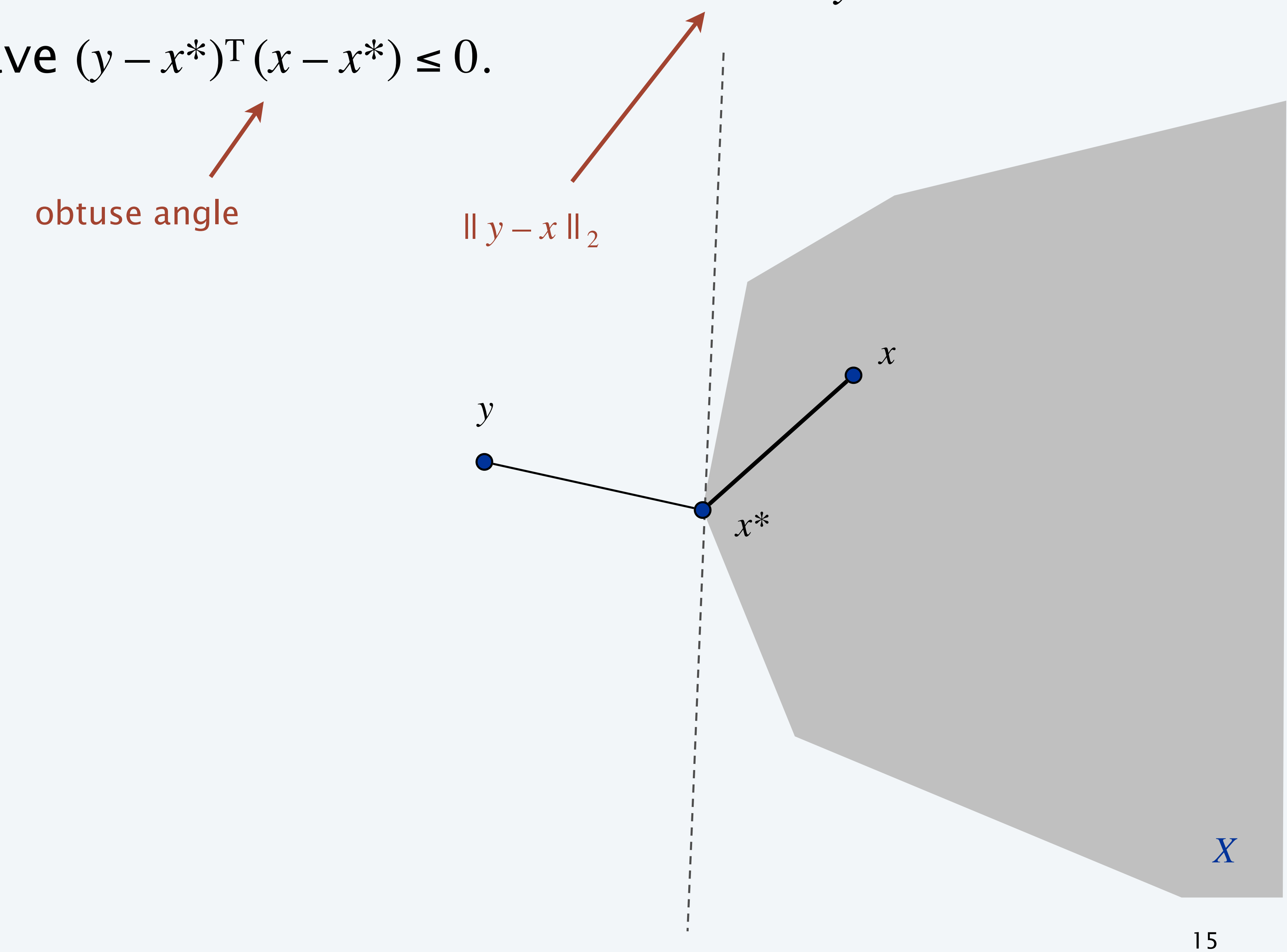
Pf. Suppose $x \in \Re^m$ is feasible for (P) and $y \in \Re^n$ is feasible for (D).

- $y \ge 0,\ Ax \le b \quad \Rightarrow \quad y^T A x \le y^T b$
- $x \ge 0,\ A^T y \ge c \quad \Rightarrow \quad y^T A x \ge c^T x$
- Combine: $c^T x \le y^T A x \le y^T b$. ▪

Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then min $\{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \le 0$.



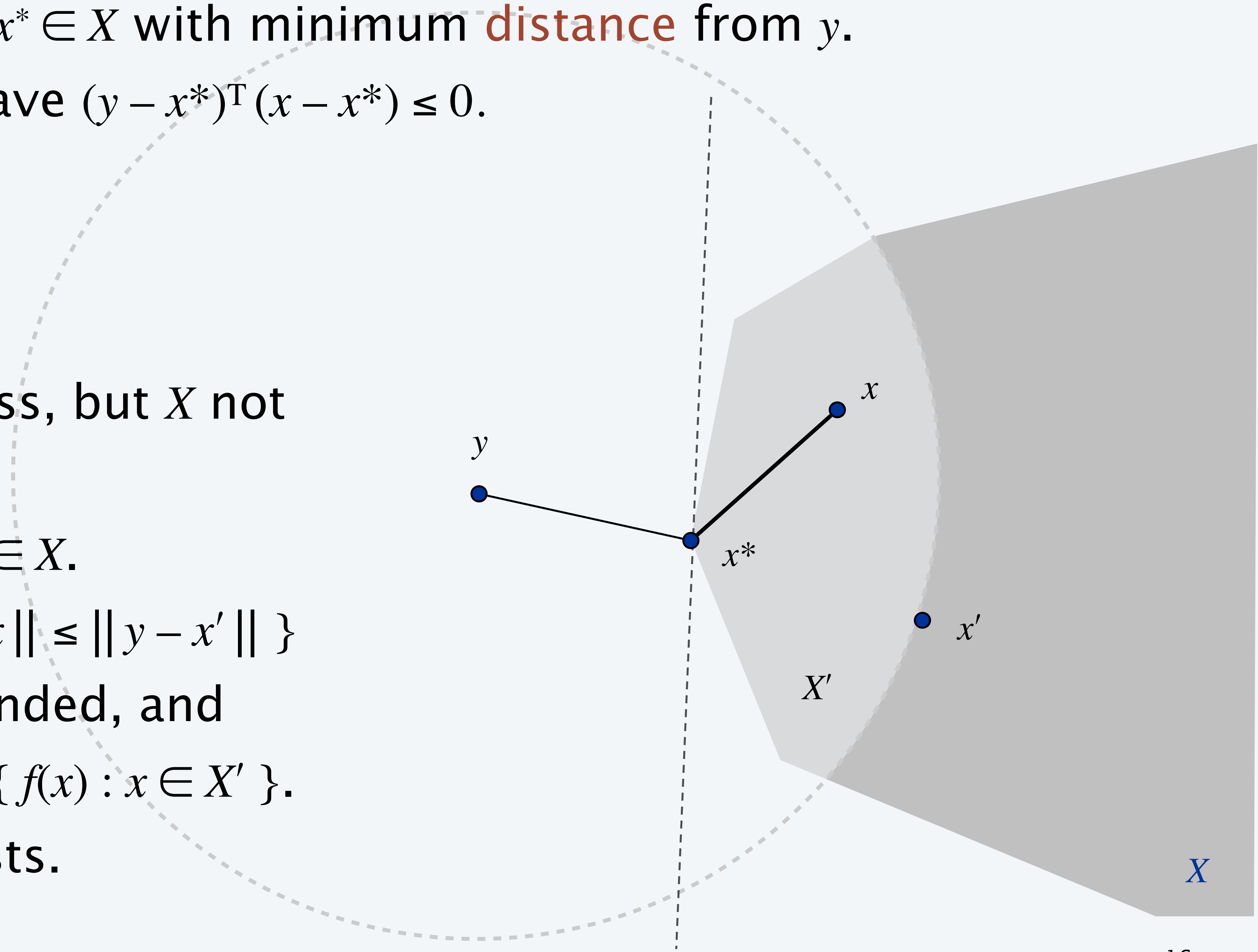
Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then min $\{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \le 0$.

Pf.

- Define $f(x) = \|y - x\|$.
- Want to apply Weierstrass, but X not necessarily bounded.
- $X \neq \phi \Rightarrow$ there exists $x' \in X$.
- Define $X' = \{ x \in X : \|y - x\| \le \|y - x'\| \}$ so that X' is closed, bounded, and min $\{ f(x) : x \in X \}$ = min $\{ f(x) : x \in X' \}$.
- By Weierstrass, min exists.



Projection lemma

Weierstrass' theorem. Let X be a compact set, and let $f(x)$ be a continuous function on X. Then min $\{ f(x) : x \in X \}$ exists.

Projection lemma. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists $x^* \in X$ with minimum distance from y. Moreover, for all $x \in X$ we have $(y - x^*)^T (x - x^*) \leq 0$.

Pf.

- x^* min distance $\Rightarrow$ $\| y - x^* \|^2 \leq \| y - x \|^2$ for all $x \in X$.
- By convexity: if $x \in X$, then $x^* + \varepsilon (x - x^*) \in X$ for all $0 < \varepsilon < 1$.
- $\| y - x^* \|^2 \leq \| y - x^* - \varepsilon (x - x^*) \|^2$

  $= \| y - x^* \|^2 + \varepsilon^2 \|(x - x^*)\|^2 - 2 \varepsilon (y - x^*)^T (x - x^*)$
- Thus, $(y - x^*)^T (x - x^*) \leq ½ \varepsilon \|(x - x^*)\|^2$.
- Letting $\varepsilon \to 0^+$, we obtain the desired result. ▪

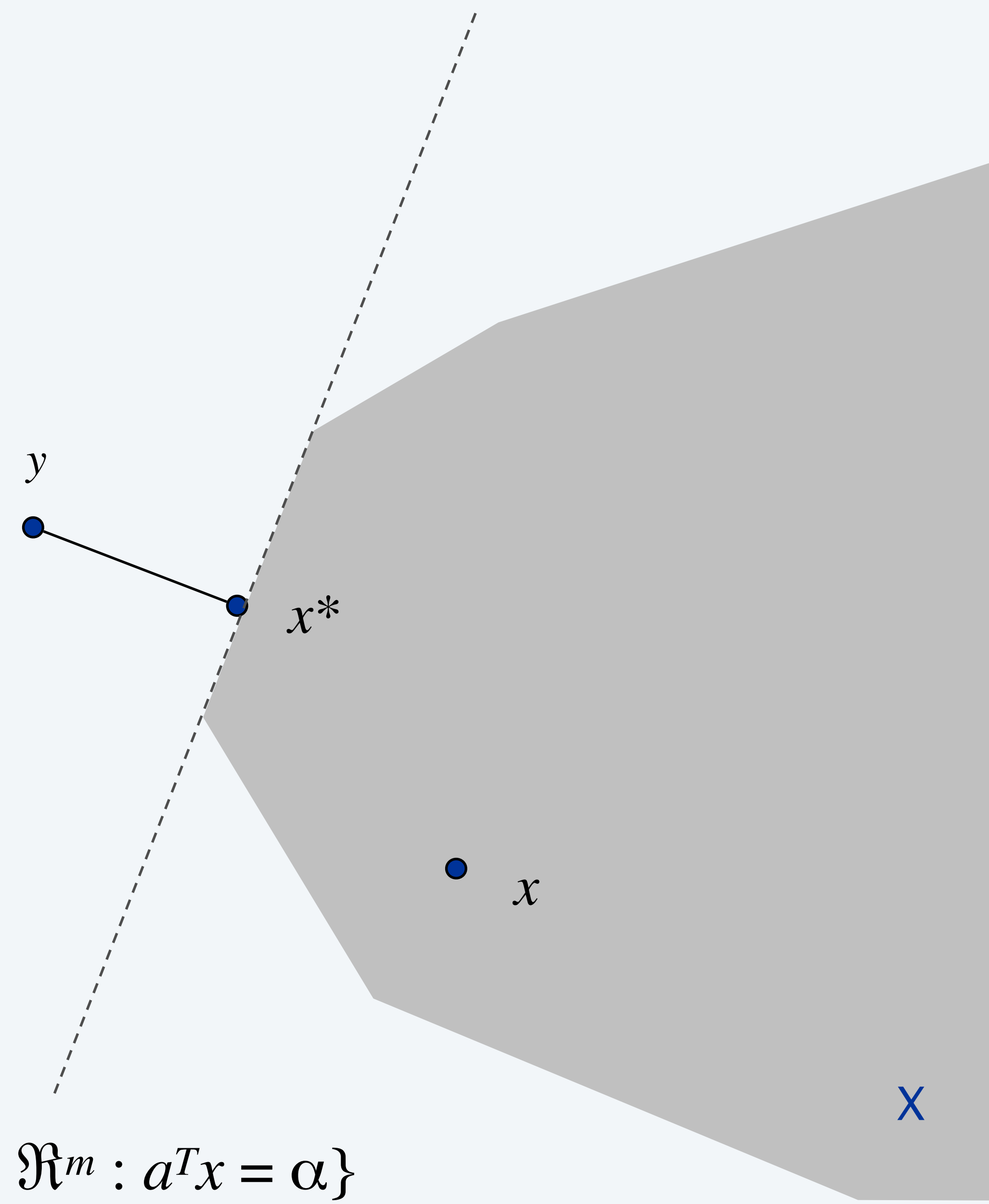
Separating hyperplane theorem

Theorem. Let $X \subset \Re^m$ be a nonempty closed convex set, and let $y \notin X$. Then there exists a hyperplane $H = \{ x \in \Re^m : a^T x = \alpha \}$ where $a \in \Re^m$, $\alpha \in \Re$ that separates y from X.

$a^T x \geq \alpha$ for all $x \in X$
$a^T y < \alpha$

Pf.

- Let x^* be closest point in X to y.
- By projection lemma,
  $(y - x^*)^T (x - x^*) \leq 0$ for all $x \in X$
- Choose $a = x^* - y \neq 0$ and $\alpha = a^T x^*$.
- If $x \in X$, then $a^T(x - x^*) \geq 0$;
  thus $\Rightarrow a^T x \geq a^T x^* = \alpha$.
- Also, $a^T y = a^T (x^* - a) = \alpha - \|a\|^2 < \alpha$ ▪



Farkas' lemma

Theorem. For $A \in \Re^{m \times n}$, $b \in \Re^m$ exactly one of the following two systems holds:

$$\text{(I)} \quad \exists\, x \in \Re^n \quad \text{s.t.} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

$$\text{(II)} \quad \exists\, y \in \Re^m \quad \text{s.t.} \quad \begin{aligned} A^T y &\geq 0 \\ y^T b &< 0 \end{aligned}$$

Pf. [not both] Suppose x satisfies (I) and y satisfies (II).
Then $0 > y^{\mathrm{T}} b = y^{\mathrm{T}} A x \geq 0$, a contradiction.

Pf. [at least one] Suppose (I) infeasible. We will show (II) feasible.

- Consider $S = \{ Ax : x \geq 0 \}$ so that S closed, convex, $b \notin S$.
- Let $y \in \Re^m, \alpha \in \Re$ be a hyperplane that separates b from S: $y^{\mathrm{T}} b < \alpha$, $y^{\mathrm{T}} s \geq \alpha$ for all $s \in S$.
- $0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^{\mathrm{T}} b < 0$
- $y^{\mathrm{T}} A x \geq \alpha$ for all $x \geq 0 \Rightarrow y^{\mathrm{T}} A \geq 0$ since x can be arbitrarily large. ▪

Another theorem of the alternative

Corollary. For $A \in \Re^{m \times n}$, $b \in \Re^{m}$ exactly one of the following two systems holds:

$$\text{(I)} \quad \exists\, x \in \Re^{n} \quad \text{s.t.} \quad Ax \le b, \quad x \ge 0$$

$$\text{(II)} \quad \exists\, y \in \Re^{m} \quad \text{s.t.} \quad A^T y \ge 0, \quad y^T b < 0, \quad y \ge 0$$

Pf. Apply Farkas' lemma to:

$$\text{(I')} \quad \exists\, x \in \Re^{n}, s \in \Re^{m} \quad \text{s.t.} \quad Ax + Is = b, \quad x, s \ge 0$$

$$\text{(II')} \quad \exists\, y \in \Re^{m} \quad \text{s.t.} \quad A^T y \ge 0, \quad Iy \ge 0, \quad y^T b < 0$$

LP strong duality

Theorem. [strong duality] For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty then max = min.

$$
\text{(P)} \quad \max c^T x \quad \text{s.t.} \quad Ax \le b, \quad x \ge 0
$$

$$
\text{(D)} \quad \min y^T b \quad \text{s.t.} \quad A^T y \ge c, \quad y \ge 0
$$

Pf. [max ≤ min] Weak LP duality.

Pf. [min ≤ max] Suppose max < α. We show min < α.

$$
\text{(I)} \quad \exists\, x \in \Re^n \quad \text{s.t.} \quad Ax \le b, \quad -c^T x \le -\alpha, \quad x \ge 0
$$

$$
\text{(II)} \quad \exists\, y \in \Re^m,\ z \in \Re \quad \text{s.t.} \quad A^T y - c\, z \ge 0, \quad y^T b - \alpha\, z < 0, \quad y,\ z \ge 0
$$

- By definition of α, (I) infeasible $\Rightarrow$ (II) feasible by Farkas' corollary.

LP strong duality

$$
\begin{aligned}
\text{(II)} \quad \exists\, y \in \Re^m,\ z \in \Re \quad & \\
\text{s.t.} \quad A^T y - cz &\geq 0 \\
y^T b - \alpha z &< 0 \\
y, z &\geq 0
\end{aligned}
$$

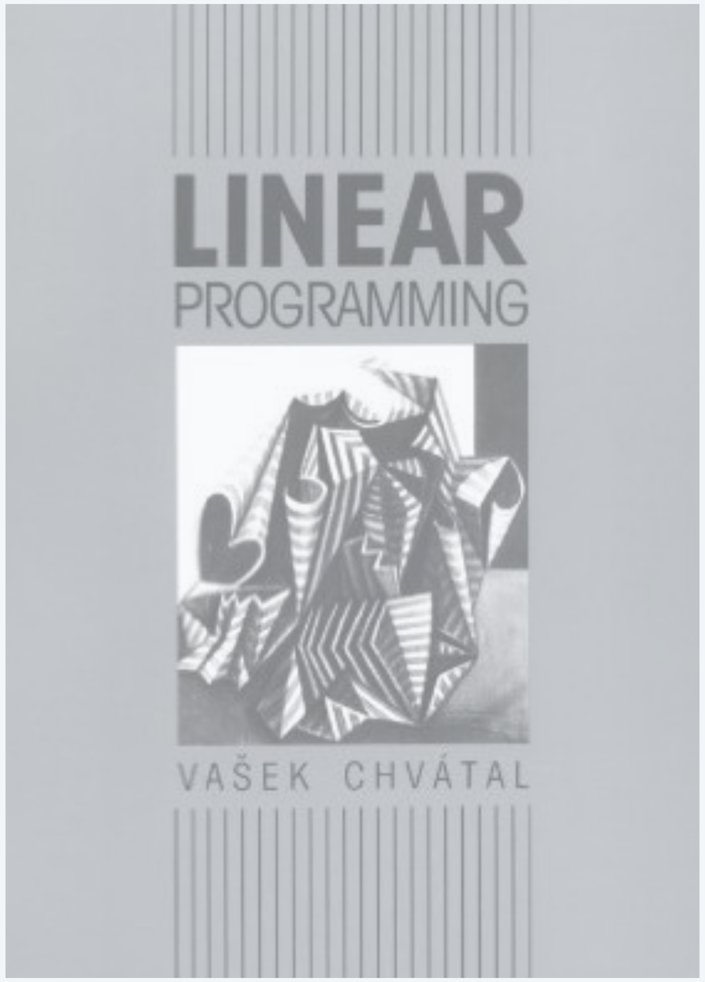
Let y, z be a solution to (II).

Case 1. $[z = 0]$

- Then, $\{\, y \in \Re^m : A^T y \geq 0,\ y^T b < 0,\ y \geq 0 \,\}$ is feasible.
- Farkas Corollary $\Rightarrow$ $\{\, x \in \Re^n : Ax \leq b, x \geq 0 \,\}$ is infeasible.
- Contradiction since by assumption (P) is nonempty.

Case 2. $[z > 0]$

- Scale y, z so that y satisfies (II) and $z = 1$.
- Resulting y feasible to (D) and $y^T b < \alpha$. ▪



Linear Programming II

- *LP duality*
- *strong duality theorem*
- *bonus proof of LP duality*
- *applications*

Strong duality theorem

Theorem. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty, then max = min.

$$
\begin{array}{llrcl}
\text{(P)} & \max c^T x & & & \\
& \text{s. t.} & Ax & = & b \\
& & x & \geq & 0
\end{array}
$$

$$
\begin{array}{llrcl}
\text{(D)} & \min y^T b & & & \\
& \text{s. t.} & A^T y & \geq & c
\end{array}
$$

Review: simplex tableaux

subtract $c_B{}^T A_B{}^{-1}$ times constraints

$$
\begin{array}{rcrcl}
c_B^T\, x_B & + & c_N^T\, x_N & = & Z \\
A_B\, x_B & + & A_N\, x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

initial tableaux

$$
\begin{array}{rcrcl}
(c_N^T & - & c_B^T\, A_B^{-1}\, A_N)\, x_N & = & Z \;-\; c_B^T\, A_B^{-1}\, b \\
I\, x_B & + & A_B^{-1}\, A_N\, x_N & = & A_B^{-1}\, b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

tableaux corresponding to basis B

multiply by $A_B{}^{-1}$

Primal solution. $x_B = A_B{}^{-1} b \geq 0,\ x_N = 0$

Optimal basis. $c_N{}^T - c_B{}^T A_B{}^{-1} A_N \leq 0$

Simplex tableaux: dual solution

subtract $c_B^T A_B^{-1}$ times constraints

$$
\begin{array}{rcl}
c_B^T x_B + c_N^T x_N & = & Z \\
A_B x_B + A_N x_N & = & b \\
x_B \;,\; x_N & \geq & 0
\end{array}
$$

initial tableaux

$$
\begin{array}{rcl}
(c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B + A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B \;,\; x_N & \geq & 0
\end{array}
$$

tableaux corresponding to basis B

multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0, \; x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Dual solution. $y^T = c^T_B A_B^{-1}$

$$
\begin{array}{rcl}
y^T b & = & c_B^T A_B^{-1} b \\
& = & c_B^T x_B + c_B^T x_B \\
& = & c^T x
\end{array}
$$

min ≤ max

$$
\begin{array}{rcl}
y^T A & = & \begin{bmatrix} y^T A_B & y^T A_N \end{bmatrix} \\
& = & \begin{bmatrix} c_B^T A_B^{-1} A_B & c_B^T A_B^{-1} A_N \end{bmatrix} \\
& = & \begin{bmatrix} c_B^T & c_B^T A_B^{-1} A_N \end{bmatrix} \\
& \geq & \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \\
& = & c^T
\end{array}
$$

dual feasible

Simplex algorithm: LP duality

subtract $c_B^T A_B^{-1}$ times constraints

$$\begin{aligned} c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B\ ,\ x_N &\geq 0 \end{aligned}$$

initial tableaux

$$\begin{aligned} (c_N^T - c_B^T A_B^{-1} A_N)\, x_N &= Z - c_B^T A_B^{-1} b \\ I\, x_B + A_B^{-1} A_N\, x_N &= A_B^{-1} b \\ x_B\ ,\ x_N &\geq 0 \end{aligned}$$

tableaux corresponding to basis B

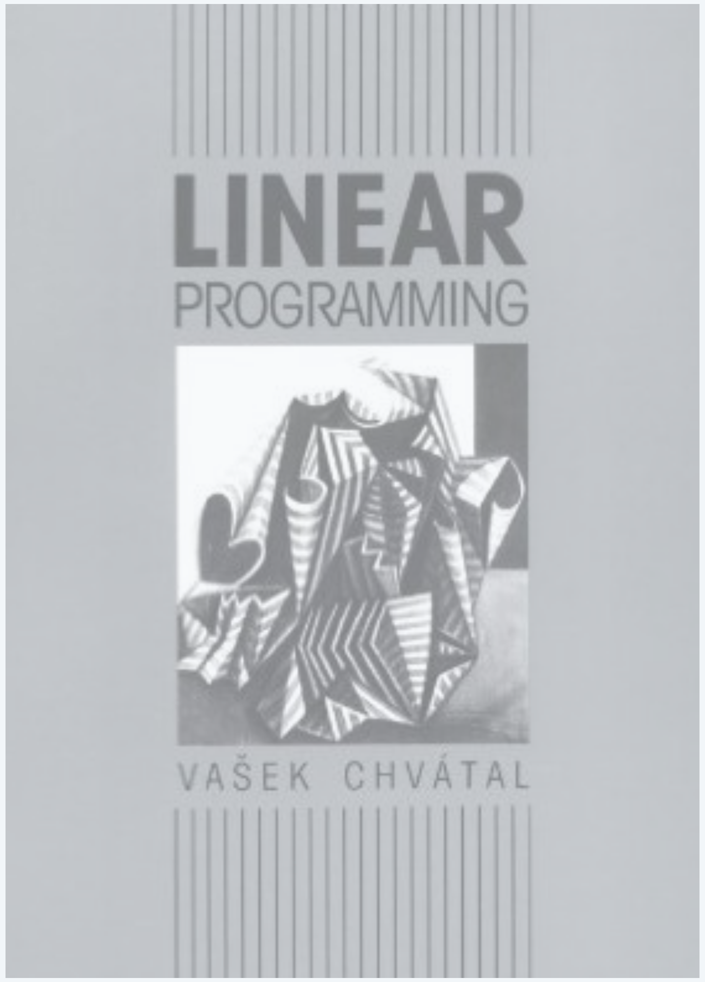
multiply by A_B^{-1}

Primal solution. $x_B = A_B^{-1} b \geq 0,\ x_N = 0$

Optimal basis. $c_N^T - c_B^T A_B^{-1} A_N \leq 0$

Dual solution. $y^T = c^T_B A_B^{-1}$

Simplex algorithm yields constructive proof of LP duality.



LINEAR PROGRAMMING II

- *LP duality*
- *strong duality theorem*
- *alternate proof of LP duality*
- *applications*

LP duality: economic interpretation

Brewer: find optimal mix of beer and ale to maximize profits.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
 & \text{s. t.} & 5A & + & 15B & \leq & 480 \\
 & & 4A & + & 4B & \leq & 160 \\
 & & 35A & + & 20B & \leq & 1190 \\
 & & A & , & B & \geq & 0
\end{array}
$$

$A^* = 12$
$B^* = 28$
$OPT = 800$

Entrepreneur: buy individual resources from brewer at min cost.

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
 & \text{s. t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
 & & 15C & + & 4H & + & 20M & \geq & 23 \\
 & & C & , & H & , & M & \geq & 0
\end{array}
$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
$OPT = 800$

LP duality. Market clears.

LP duality: sensitivity analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

A. corn \$1, hops \$2, malt \$0.

Q. Suppose a new product “light beer” is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

A. At least 2 (\$1) + 5 (\$2) + 24 (\$0) = \$12 / barrel.

LP is in NP ∩ co-NP

LP. For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, $\alpha \in \Re$, does there exist $x \in \Re^n$ such that: $Ax = b,\ x \geq 0,\ c^T x \geq \alpha$?

Theorem. LP is in **NP** ∩ co-**NP**.

Pf.

- Already showed LP is in **NP**.
- If LP is infeasible, then apply Farkas' lemma to get certificate of infeasibility:

$$
\begin{array}{lrrcl}
\text{(II)} & \exists\, y \in \Re^m,\ z \in \Re & & & \\
 & \text{s.t.} & A^T y & \geq & 0 \\
 & & y^T b \,-\, \alpha z & < & 0 \\
 & & z & \geq & 0
\end{array}
$$

or equivalently, $y^T b - \alpha z = -1$