CS7800: Advanced Algorithms

Lecture 23: Regret minimization

- Jon's favorite algorithm
- Application to zero-sum games

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What do these problems have to do with each other?

1. Finding equilibria in zero-sum games
2. Solving linear programs
3. Minimizing a convex function $\ell(\theta)$
4. Boosting weak learning algorithms to better ensembles
5. Approximating SETCOVER
6. Generating privacy preserving synthetic data
7. Proving Chernoff bounds
8. Training many models without overfitting

No-Regret Learning

- Set of actions $\{1, 2, \ldots, n\}$ you can take each day
- Play for T days
- For $t = 1, 2, \ldots, T$:
  - You choose $p^t = (p_1^t, \ldots, p_n^t)$ $\left[\sum_{i=1}^{n} p_i^t = 1,\ p_i^t \geq 0\right]$
  - You observe losses $\ell^t = (\ell_1^t, \ldots, \ell_n^t)$ $[\ell_i^t \in [0,1]]$
  - You suffer loss $\sum_{i=1}^{n} \ell_i^t p_i^t \in [0,1]$

Regret $R = \underbrace{\sum_{t=1}^{T} \sum_{i=1}^{n} p_i^t \ell_i^t}_{L = \sum_{t=1}^{T} L^t} - \underbrace{\min_{j \in [n]} \sum_{t=1}^{T} \ell_j^t}_{L^*}$

How to play when one action has no loss

- Initialize $w_i^1 = 1$ for $i = 1, \ldots, n$

  $W^1 = \sum_{i=1}^{n} w_i^1 = n$

- For $t = 1, 2, \ldots, T$:
  - Play strategy $p^t = \frac{w^t}{W^t}$
  - Receive losses $\ell^t \in \{0,1\}^n$ and suffer loss $\underbrace{\sum_{i=1}^{n} p_i^t \ell_i^t}_{= L^t}$
  - Update: set $w_i^{t+1} = \begin{cases} 0 & \text{if } \ell_i^t = 1 \\ w_i^t & \text{if } \ell_i^t = 0 \end{cases}$

Worst Example

actions

time	$\boxed{1}$	2	3	...			n	
1	0	0	0	...	0	1 ...	1	lose 1/2
2	0	0	0	1 ... 1				lose 1/2
3	0	0	1 1					lose 1/2
⋮		⋮						
	0							lose 1/2

$$\left.\begin{array}{l} \text{lose } 1/2 \\ \text{lose } 1/2 \\ \text{lose } 1/2 \\ \vdots \\ \text{lose } 1/2 \end{array}\right\} \log_2 n$$

$$\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{2}$$

Analysis

- Initialize $w_i^1 = 1$ for $i = 1, \dots, n$
  $W^1 = \sum_{i=1}^{n} w_i^1 = n$
- For $t = 1, 2, \dots, T$:
  - Play strategy $p^t = \frac{w^t}{W^t}$
  - Receive losses $\ell^t \in \{0,1\}^n$ and suffer loss $\overbrace{\sum_{i=1}^{n} p_i^t \ell_i^t}^{= L^t}$
  - Update: set $w_i^{t+1} = \begin{cases} 0 & \text{if } \ell_i^t = 1 \\ w_i^t & \text{if } \ell_i^t = 0 \end{cases}$

① $W^1 = n$

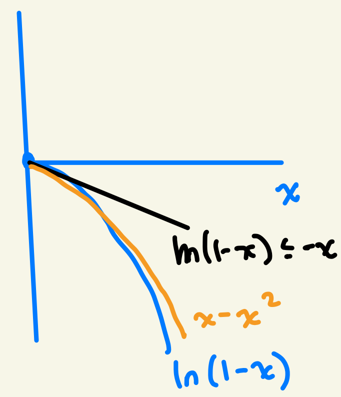
② $W^T \geq 1$

③ $$\frac{W^t}{W^{t-1}} = (1 - L^{t-1})$$

$$1 \leq W^T = n \cdot \prod_{t=1}^{T} \frac{W^t}{W^{t-1}} = n \cdot \prod_{t=1}^{T} (1 - L^{t-1})$$

$$0 \leq \ln(n) + \sum_{t=1}^{T} \ln(1 - L^t) \leq \ln(n) - \sum_{t=1}^{T} L^t$$

$$\sum_{t=1}^{T} L^t \leq \ln(n)$$



Multiplicative Weights

- Initialize $w_i^1 = 1$ for $i = 1, \dots, n$

  $W^1 = \sum_{i=1}^{n} w_i^1 = n$

- For $t = 1, 2, \dots, T$:
  - Play strategy $p^t = \frac{w^t}{W^t}$
  - Receive losses ℓ^t and suffer loss $\underbrace{\sum_{i=1}^{n} p_i^t \ell_i^t}_{= L^t}$
  - Update $w_i^{t+1} = (1-\eta)^{\ell_i^t} \cdot w_i^t$

    multiplicative weights updates

    $W^{t+1} = \sum_{i=1}^{n} w_i$

- Initialize $w_i^1 = 1$ for $i = 1, \dots, n$

  $W^1 = \sum_{i=1}^{n} w_i^1 = n$

- For $t = 1, 2, \dots, T$:
  - Play strategy $p^t = \frac{w^t}{W^t}$
  - Receive losses ℓ^t and suffer loss $\overbrace{\sum_{i=1}^{n} p_i^t \ell_i^t}^{= L^t}$
  - Update $w_i^{t+1} = (1-\eta)^{\ell_i^t} \cdot w_i^t$

    $W^{t+1} = \sum_{i=1}^{n} w_i$

Analysis

① $W^1 = n$

② For every j: $W^T \geq w_j^T = \prod_{t=1}^{T} (1-\eta)^{\ell_j^t}$

③
$$W^t = \sum_{i=1}^{n} w_i^t = \sum_{i=1}^{n} w_i^{t-1} \cdot (1-\eta)^{\ell_i^{t-1}} \leq \sum_{i=1}^{n} w_i^{t-1} \cdot (1 - \eta \ell_i^{t-1})$$
$$= \sum_{i=1}^{n} w_i^{t-1} - \eta w_i^{t-1} \ell_i^{t-1}$$
$$= \sum_{i=1}^{n} w_i^{t-1} - W^{t-1} \cdot \eta p_i^{t-1} \ell_i^{t-1}$$
$$= W^{t-1} - W^{t-1} \eta L^{t-1}$$
$$= W^{t-1} (1 - \eta L^{t-1})$$

$$\frac{W^t}{W^{t-1}} \leq 1 - \eta L^{t-1}$$

Analysis

① $W^1 = n$

② For every j : $W^T \geq w_j^T = \prod_{t=1}^{T} (1-\eta)^{\ell_j^t}$

- Initialize $w_i^1 = 1$ for $i = 1, \ldots, n$

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- For $t = 1, 2, \ldots, T$:
  - Play strategy $p^t = \frac{w^t}{W^t}$
  - Receive losses ℓ^t and suffer loss $\overbrace{\sum_{i=1}^{n} p_i^t \ell_i^t}^{= L^t}$
  - Update $w_i^{t+1} = (1-\eta)^{\ell_i^t} \cdot w_i^t$

    $W^{t+1} = \sum_{i=1}^{n} w_i$

③ $\frac{W^t}{W^{t-1}} \leq 1 - \eta L^{t-1}$

$$\prod_t (1-\eta)^{\ell_{j^*}^t} \leq \cancel{W^T} \leq n \cdot \prod_t 1 - \eta L^{t-1}$$

$$L^*(-\eta - \eta^2) \leq \sum_t \ell_{j^*}^t \ln(1-\eta) \leq \ln(n) + \sum_t \ln(1 - \eta L^{t-1}) \leq \ln(n) - \eta L$$

$$-\eta L^* - \eta^2 L^* \leq \ln(n) - \eta L$$

$$\eta(L - L^*) \leq \ln(n) + \eta^2 L^*$$

$$L - L^* \leq \frac{\ln(n)}{\eta} + \eta L^* \leq \frac{\ln(n)}{\eta} + \eta T$$

$$\eta = \sqrt{\frac{\ln(n)}{T}} \Rightarrow L - L^* \leq 2\sqrt{T \ln(n)}$$