CS7800: Advanced Algorithms

Lecture 11: Linear Programming II

- Duality
- The Ellipsoid Algorithm

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Geometry of Linear Programs

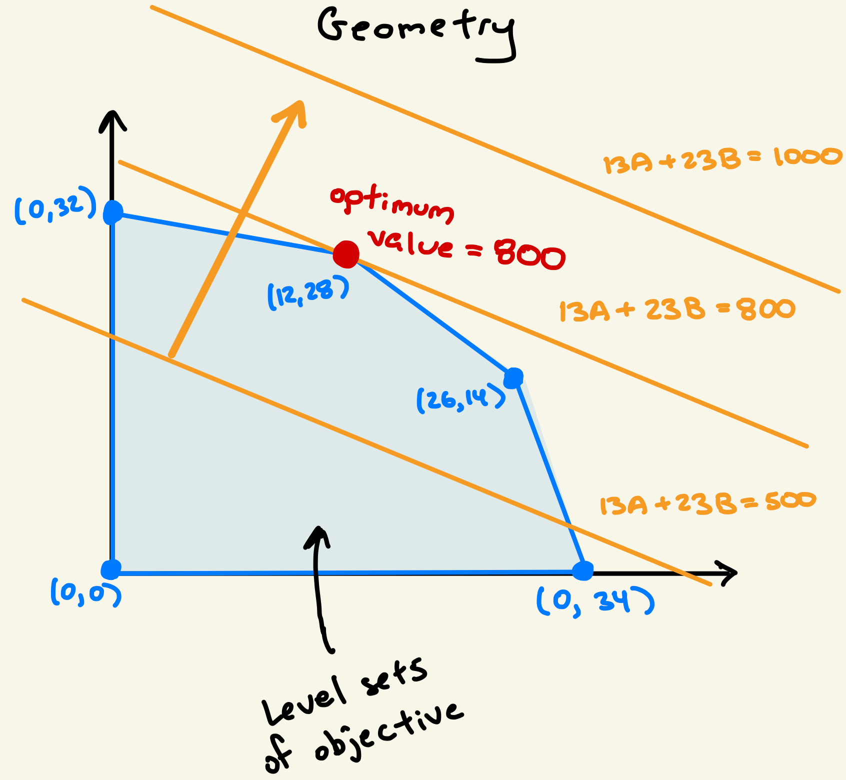
Algebra

$$
\begin{aligned}
\max \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \le 480 \\
& 4A + 4B \le 160 \\
& 35A + 20B \le 1190 \\
& A, B \ge 0
\end{aligned}
$$

opt solution = (12, 28)

opt value = 800

Geometry



How do we know we found an optimal solution?

Find an upper bound on opt

optimization problem

$$\begin{aligned} \max\quad & 13A + 23B \\ \text{s.t.}\quad & 5A + 15B \le 480 \\ & 4A + 4B \le 160 \\ & 35A + 20B \le 1190 \\ & A, B \ge 0 \end{aligned}$$

optimal value = 800

3x

$$15A + 45B \le 1440$$

6x

$$13A + 23B \le 24A + 24B \le 960$$

How do we know we found an optimal solution?

Find an upper bound on opt

optimization problem

$$\begin{aligned} \max \quad & 13A + 23B \\ \text{s.t.} \quad & 5A + 15B \le 480 \\ & 4A + 4B \le 160 \\ & 35A + 20B \le 1190 \\ & A, B \ge 0 \end{aligned}$$

optimal value = 800

$1\times$

$2\times$

$$\begin{array}{rl} & 5A + 15B \le 480 \\ + & 8A + 8B \le 320 \\ \hline & 13A + 23B \le 800 \end{array}$$

Derive an upper bound on the optimal value by combining constraints.

Rules:
- coefficient on each constraint ≥ 0
- coefficient on each variable greater than that of the objective

The Dual of a Linear Program

primal (P)
optimization problem

$$\begin{aligned} \max \quad & 13A + 23B \\ \text{s.t.} \quad & 5A + 15B \le 480 \quad (C) \\ & 4A + 4B \le 160 \quad (H) \\ & 35A + 20B \le 1190 \quad (M) \\ & A, B \ge 0 \end{aligned}$$

optimal value = 800

dual (D)
optimization problem

$$\begin{aligned} \min_{C,H,M} \quad & 480C + 160H + 1190M \\ & 5C + 4H + 35M \ge 13 \\ & 15C + 4H + 20M \ge 23 \\ & C, H, M \ge 0 \end{aligned}$$

optimal value = 800

- Any feasible solution to D gives an upper bound on P
- Any feasible solution to P gives a lower bound on D

The Dual of a Linear Program

primal (P)
optimization problem

$$\max_x \; c^T x$$
$$\text{s.t.} \quad Ax \le b \qquad (y \in \mathbb{R}^m)$$
$$x \ge 0$$

dual (D)
optimization problem

$$\min_y \; y^T b$$
$$\text{s.t.} \quad A^T y \ge c$$
$$y \ge 0$$

$$[- \, y^T \, -] \begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_m & - \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

Weak Duality

$$c^T x \le y^T A x \le y^T b$$

The Dual of a Linear Program

<u>Fact</u>. Can take a dual of an arbitrary LP using the following recipe

Primal (P)	maximize	minimize	Dual (D)
constraints	$a\,x = b_i$ $a\,x \leq b$ $a\,x \geq b_i$	y_i unrestricted $y_i \geq 0$ $y_i \leq 0$	variables
variables	$x_j \geq 0$ $x_j \leq 0$ unrestricted	$a^{\mathrm{T}}y \geq c_j$ $a^{\mathrm{T}}y \leq c_j$ $a^{\mathrm{T}}y = c_j$	constraints

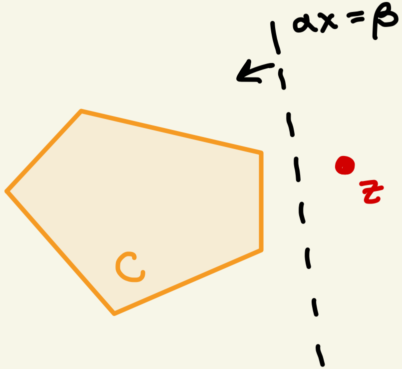
<u>Fact</u>: The dual of the dual is the primal

Linear Programming Duality

<u>Thm:</u> If the primal and dual are both feasible and bounded then the value of the primal and dual are equal.

Strong Duality: Proof Sketch

Idea #1: Separating Hyperplane Theorem



Thm: If $C \subseteq \mathbb{R}^n$ is a closed, convex set and $z \in \mathbb{R}^n$ is any point not in C, there exists $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ s.t.

(1) $\alpha^T x \geq \beta$ for all $x \in C$ (2) $\alpha^T z < \beta$

Strong Duality : Proof Sketch

<u>Idea #2:</u> (Farkas' Lemma) Remains to prove that given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following is true:

(1) There is an $x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$

(2) There exists $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ and $y^T b < 0$

$$\boxed{\begin{array}{l} y^T A x \geq 0 \text{ for any } x \geq 0 \\ y^T b < 0 \end{array} \Rightarrow \nexists\, x \geq 0 \text{ s.t. } Ax = b}$$

Interesting Stmt: $\neg(1) \Rightarrow (2)$

Strong Duality: Proof Sketch

Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following is true:

(1) There is an $x \in \mathbb{R}^n$ s.t. $x \geq 0$ and $Ax = b$

(2) There exists $y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ and $y^T b < 0$

<u>"Proof"</u>:

- Let $Q = \{ w : \exists\, x \geq 0 \text{ s.t. } Ax = w \}$
- If (1) is false then $b \notin Q$, so there exists $\alpha \in \mathbb{R}^m$ s.t.

  $\alpha^T w \geq 0$ for all $w \in Q$ $\qquad$ $\alpha^T b < 0$

  small trick

- Goal is to show that setting $y = \alpha$ satisfies (2)
  - $y^T b = \alpha^T b < 0$ (easy)
  - $y^T A = \alpha^T A \geq 0$ (harder)

    $(\alpha^T A)_j$ is $\alpha \cdot \underbrace{(j^{th} \text{ col of } A)}_{\text{elt of } Q}$

    therefore $\alpha^T A_j \geq 0$ for all j

Strong Duality : Proof Sketch

Idea #3: Apply Farkas' Lemma

$$\boxed{\begin{array}{lll} (P) & \max\limits_{x} & c^T x \\ & \text{s.t.} & Ax = b \\ & & x \geq 0 \end{array}}$$

v^* is the opt value

Application: Zero-Sum Games and the Minimax Thm

- Two players Colin and Rowena
- Payoff matrix $A \in \mathbb{R}^{m \times n}$

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & +1 \\ P & +1 & 0 & -1 \\ S & -1 & +1 & 0 \end{array}$$

Rowena plays i, Colin plays j $\Rightarrow$ Rowena gets A_{ij}, Colin gets $-A_{ij}$

- Randomized strategies

Rowena: $r = (r_1 \ldots r_m)$ $\sum_i r_i = 1$ $r_i \geq 0$

Colin: $c = (c_1 \ldots c_n)$ $\sum_j c_j = 1$ $c_j \geq 0$

$r^T A c$ = expected payoff to Rowena

- How would Colin / Rowena play if he/she went first?

$$\underbrace{\max_r \min_c r^T A c}_{\text{Rowena goes first}} \leq \underbrace{\min_c \max_r r^T A c}_{\text{Colin goes first}}$$

Application: Zero-Sum Games and the Minimax Thm

Minimax Theorem: $$\underbrace{\max_x \left(\min_y x^T A y \right)}_{\text{Rowena goes first}} = \underbrace{\min_y \left(\max_x x^T A y \right)}_{\text{Colin goes first}}$$

Proof Sketch via LP Duality:

① $$\max_x \left(\min_y x^T A y \right) = \max_x \left(\min_j x^T \cdot (j^{th} \text{ col of } A) \right)$$

$$= \max_x \left(\min_j \sum_{i=1}^{m} x_i A_{ij} \right)$$

Application: Zero-Sum Games and the Minimax Thm

Minimax Theorem: $\underbrace{\max_x \left(\min_y x^T A y\right)}_{\text{Rowena goes first}} = \underbrace{\min_y \left(\max_x x^T A y\right)}_{\text{Colin goes first}}$

Proof Sketch via LP Duality:

Rowena's view

① $\max_x \left(\min_y x^T A y\right) = \max_x \left(\min_j \sum_{i=1}^m x_i A_{ij}\right)$

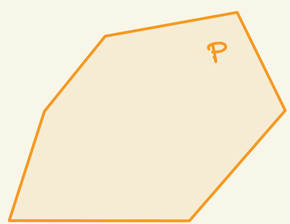
② Write an LP

$$\begin{aligned} \max_{r,x} \quad & r \\ \text{s.t.} \quad & \sum_{i=1}^m x_i A_{ij} \geq r \quad \text{for } j = 1, \dots, n \\ & \sum_{i=1}^m x_i = 1 \\ & x_i \geq 0 \quad \text{for } i = 1, \dots, m \end{aligned}$$

③ Show that Rowena and Colin have dual LPs!

Ellipsoid Algorithm

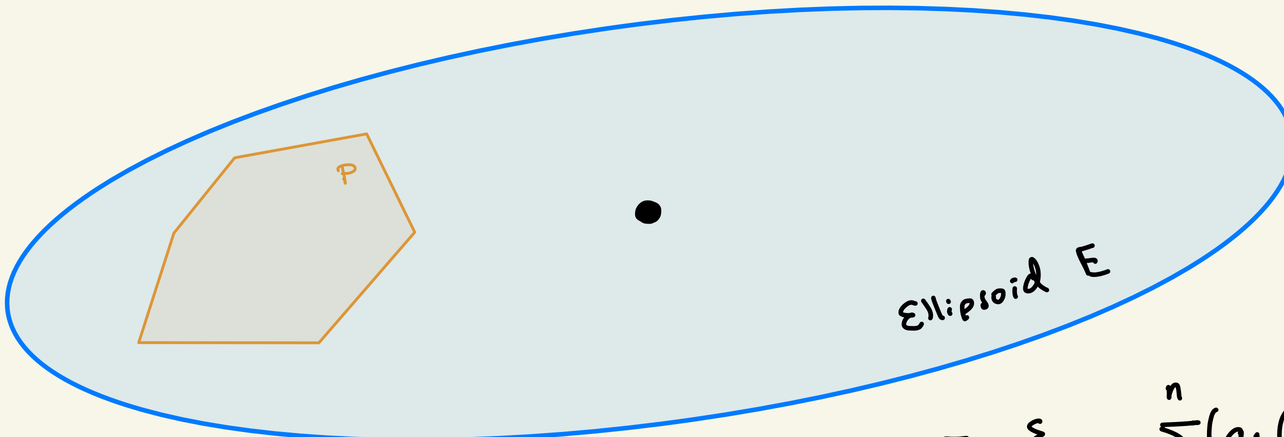
Can we solve linear programs in polynomial time?



Enough to find a feasible point

Ellipsoid Algorithm

① Find an ellipsoid containing P. How?

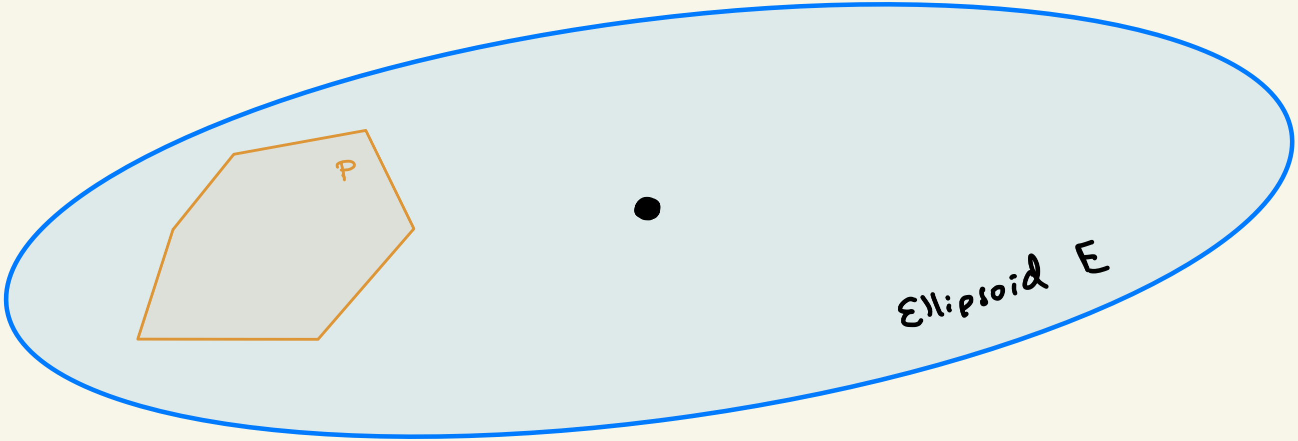


$$E = \{x : \sum_{i=1}^{n} (a_i(x_i - c_i))^2 \le 1\}$$

Enough to find a feasible point

Ellipsoid Algorithm

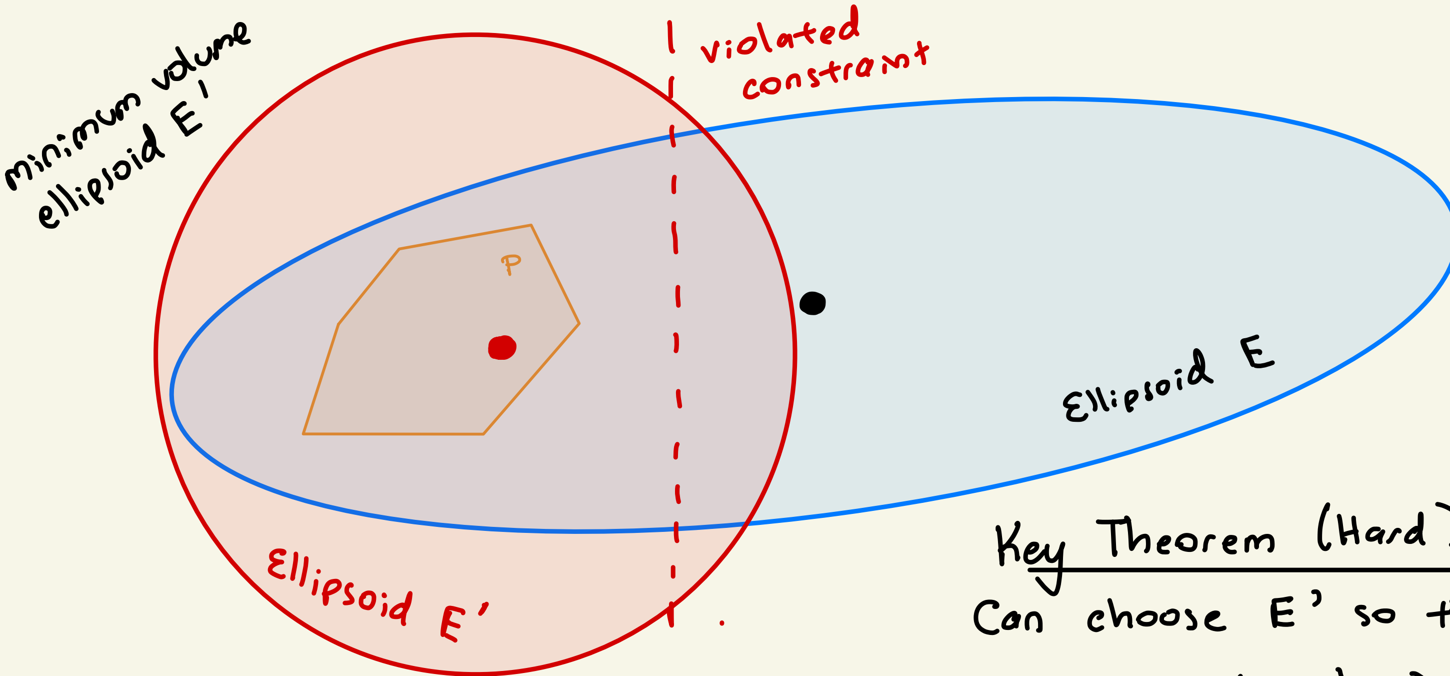
② Check if the center of the ellipsoid is feasible. How?



Enough to find a feasible point

Ellipsoid Algorithm

③ Find a violated constraint and use it to improve the ellipsoid

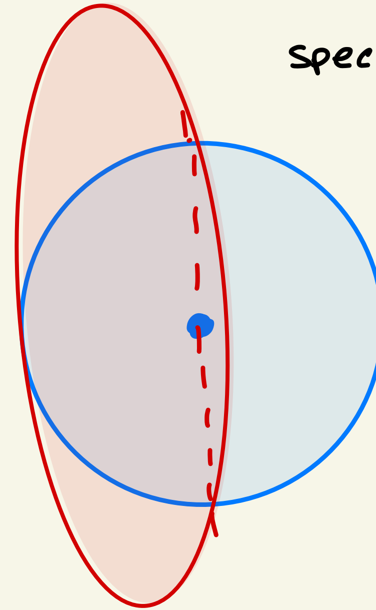


Key Theorem (Hard):

Can choose E' so that

$$\mathrm{vol}(E') \approx \left(1 - \frac{1}{2(n+1)}\right)\mathrm{vol}(E)$$

Ellipsoid Algorithm



Key Theorem (Hard):

Can choose E' so that

$$\mathrm{vol}(E') \approx \left(1 - \frac{1}{2(n+1)}\right)\mathrm{vol}(E)$$

$$E = \left\{ x : \sum_i x_i^2 \leq 1 \right\}$$

$$E' = \left\{ x : \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{n+1} - 1\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i \leq 1 \right\}$$

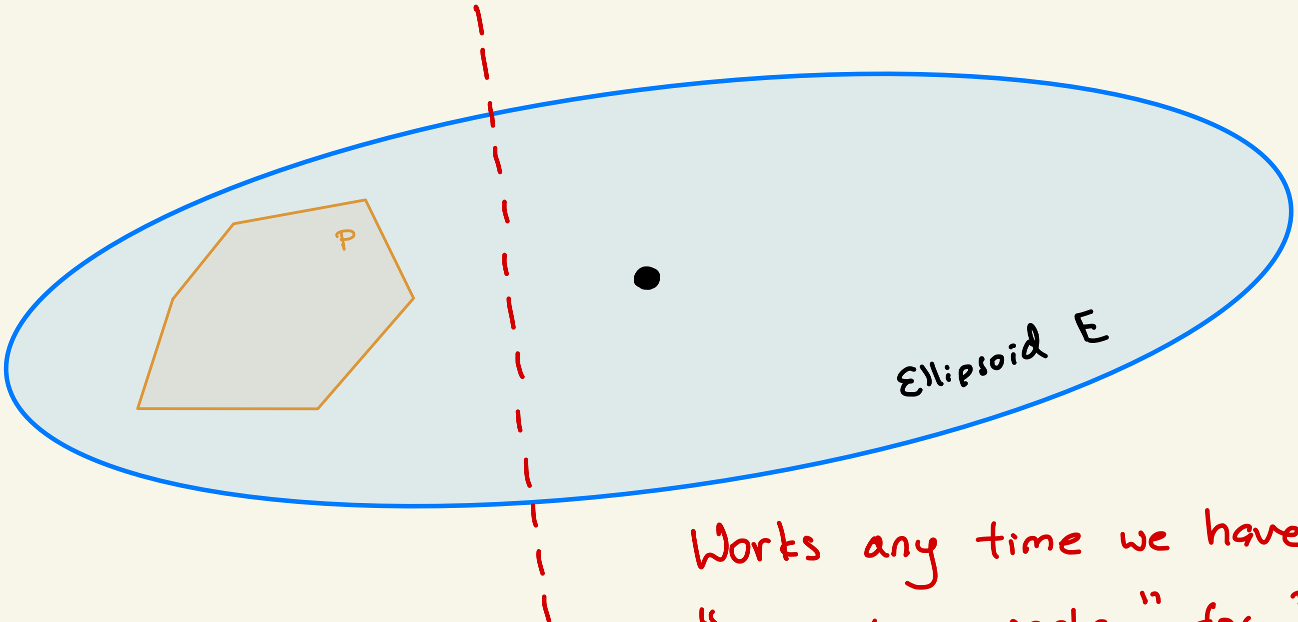
$$\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} = \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} \left(\frac{n}{n+1}\right)$$

$$= \left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \left(1 - \frac{1}{n+1}\right)$$

$$\approx e^{\left(\frac{1}{n^2-1}\right)\left(\frac{n-1}{2}\right)} e^{\frac{-1}{n+1}} = e^{\frac{-1}{2(n+1)}} \approx 1 - \frac{1}{2(n+1)}$$

Ellipsoid Algorithm

2. Check if the center of the ellipsoid is feasible. How?



Works any time we have a "separation oracle" for P!

Enough to find a feasible point