CS7800: Advanced Algorithms

Lecture 10: Linear Programming I

- Concepts
- Simplex Algorithm
- Duality

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Our Favorite Linear Program

How can our brewery maximize profits?

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

- 34 ale, 0 beer $\Rightarrow$ \$442
- 0 ale, 32 beer $\Rightarrow$ \$736
- 7.5 ale, 29.5 beer $\Rightarrow$ \$776
- 12 ale, 28 beer $\Rightarrow$ \$800

$$\max_{A,B} \quad 13A + 23B$$

$$5A + 15B \le 480$$

$$4A + 4B \le 160$$

$$35A + 20B \le 1190$$

$$A, B \ge 0$$

Linear Programming

Optimize a linear objective subject to linear inequalities

decision variables

objective function

constraints

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n} \quad & \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} a_{ij} x_i = b_j \qquad 1 \le j \le m \\
& x_i \geq 0 \qquad 1 \le i \le n
\end{aligned}
$$

(all vectors are col.) $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n} \quad & c^T x \\
\text{s.t.} \quad & Ax \le b \\
& x \geq 0
\end{aligned}
$$

Standard Form LPs

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^{n} c_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} a_{ij} x_i = b_j \qquad 1 \le j \le m$$

$$x_i \geq 0 \qquad 1 \le i \le n$$

Equality to Inequality

$$a^T x = b \implies \begin{matrix} a^T x \le b \\ a^T x \geq b \end{matrix}$$

Inequalities to Equalities

$$a^T x \le b \implies a^T x + s = b \qquad s \geq 0$$

Min to Max

$$\min c^T x \implies \max -c^T x$$

Unconstrained to Non-negative

$$x_i \implies x_i^+, x_i^- \geq 0$$

$$x_i = x_i^+ - x_i^-$$

Our Favorite Linear Program (in Standard form)

How can our brewery maximize profits?

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

decide how much is left over

$$\begin{aligned}
\max\quad & 13A + 23B \\
\text{s.t.}\quad & 5A + 15B + S_C = 480 \\
& 4A + 4B + S_H = 160 \\
& 35A + 20B + S_M = 1190 \\
& A, B, S_C, S_H, S_M \geq 0
\end{aligned}$$

Some Examples of Linear Programs

Maximum Flow

$G = (V, E, \{c(e)\}, s, t)$

flow $\{f(e)\}$

objective

decision vars

$$\max_{f} \sum_{e \text{ out of } s} f(e)$$

constraints

$$\text{s.t.} \quad \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = 0 \quad \forall v \notin \{s, t\} \quad \text{(conservation)}$$

$$f(e) \leq c(e) \quad \text{(capacity)}$$

$$f(e) \geq 0 \quad \text{(non-neg)}$$

Some Examples of Linear Programs

Minimum Cost Flow

$G = (V, E, \{c(e)\}, s, t)$ a flow $d \geq 0$, edge costs $\{\$(e)\}$

flow $\{f(e)\}$

$$\min \sum_e \$(e) \cdot f(e)$$

$$\sum_{e \text{ out of } s} f(e) \geq d$$

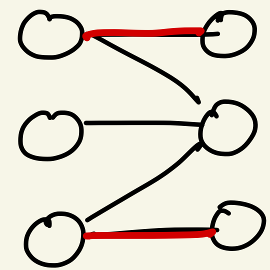
s.t. $\displaystyle\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = 0 \quad \forall v \notin \{s, t\}$ (conservation)

$f(e) \leq c(e)$ (capacity)

$f(e) \geq 0$ (non-neg)

Some Examples of Linear Programs

Bipartite Matching



$$\max_{\{x(e)\}} \sum_{e} x(e)$$

$$\text{s.t. } \forall v \quad \sum_{e \text{ incident on } v} x(e) \le 1$$

$$\boxed{x(e) \in \{0,1\}} \Rightarrow 0 \le x(e) \le 1$$

not a linear program

Dealing w/ non-integrality:

- Prove opt solution is integral
- Round to an integral sol

Geometry of Linear Programs

Algebra

$$\max \; 13A + 23B$$

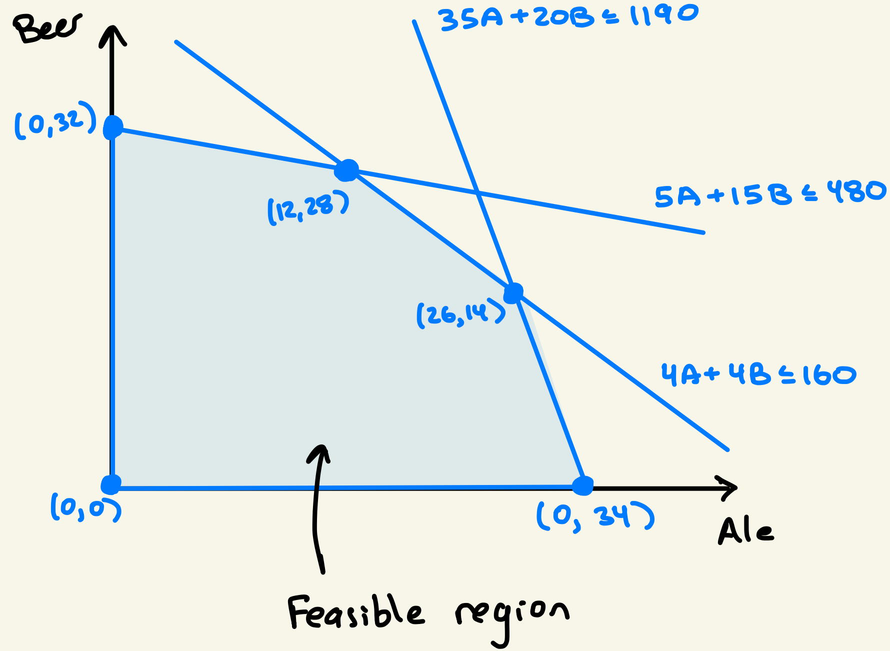
$$\text{s.t.} \quad 5A + 15B \le 480$$

$$4A + 4B \le 160$$

$$35A + 20B \le 1190$$

$$A, B \ge 0$$

Geometry

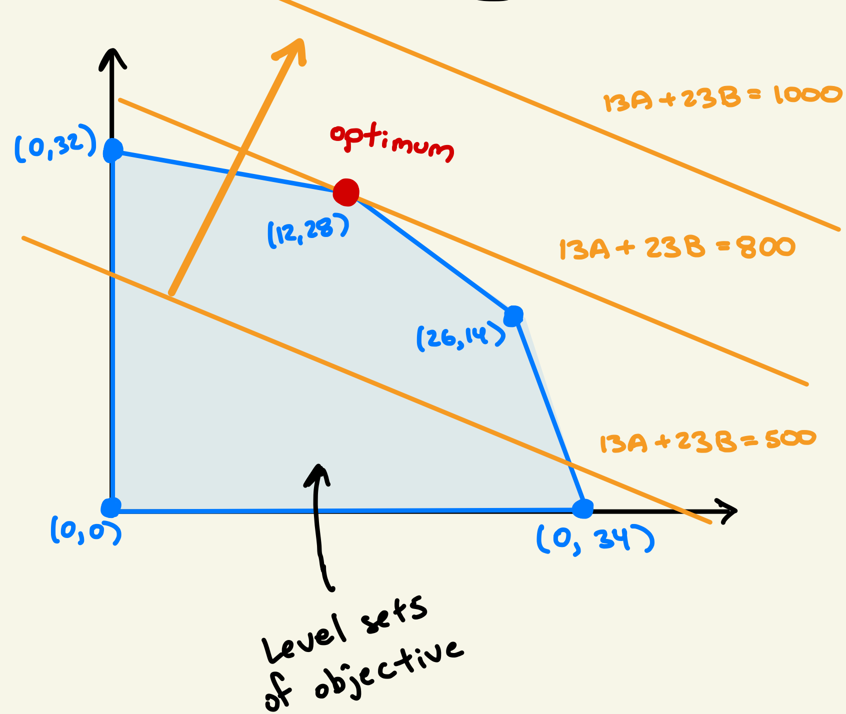


Geometry of Linear Programs

Algebra

$$\begin{aligned}
\max \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B \le 480 \\
& 4A + 4B \le 160 \\
& 35A + 20B \le 1190 \\
& A, B \ge 0
\end{aligned}$$

Geometry



Geometry of Linear Programs

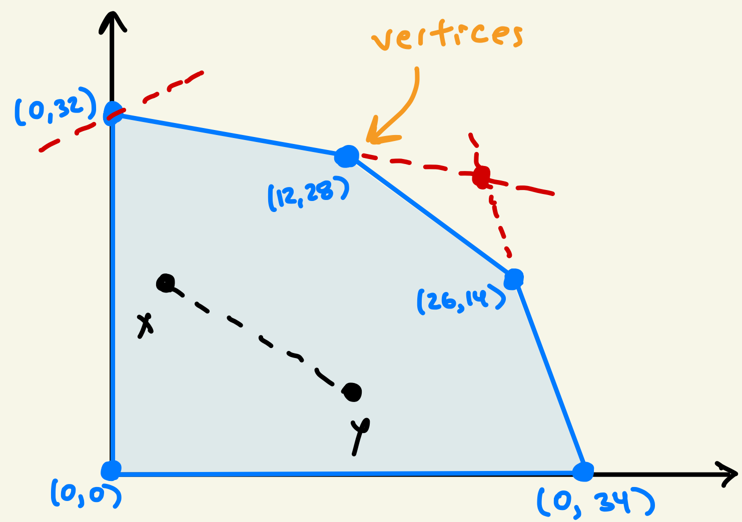
Convexity

A set P is <u>convex</u> if

$$\alpha x + (1-\alpha) y \in P$$

for every $x, y \in P$ and $0 \le \alpha \le 1$

A <u>vertex</u> is a point v that is not a convex combination of two distinct $x, y \in P$



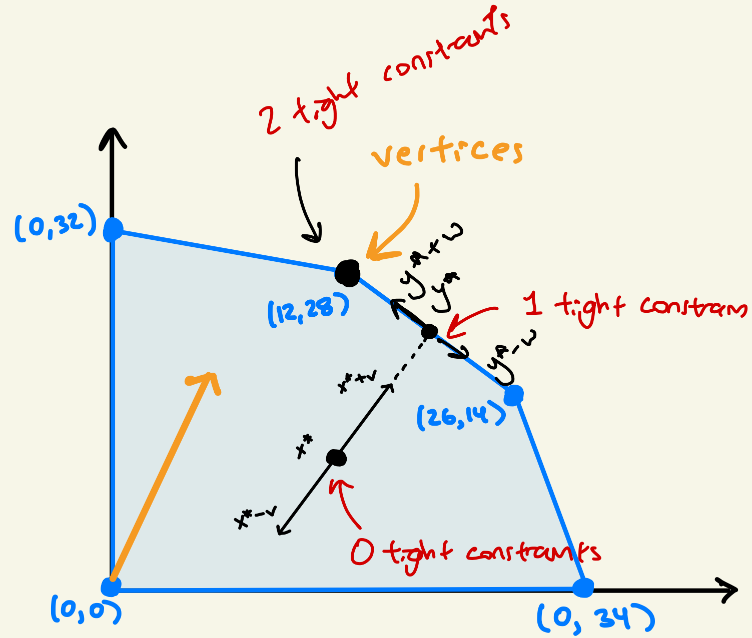
vertices are feasible points where $\geq$ $\underset{\sim}{n}$ non-dependent constraints intersect

\# of decision

Geometry of Linear Programs

Theorem: If the LP has an optimal solution, then it has an optimal solution at a vertex

We can restrict our algorithm to search for vertizes

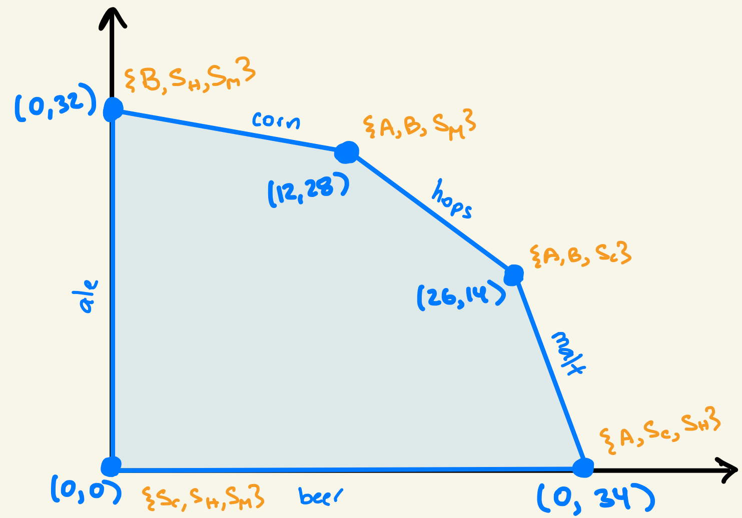


Geometry of Linear Programs

$$
\begin{array}{lrl}
\max & 13A + 23B & \\
\text{s.t.} & 5A + 15B \le 480 & C \\
 & 4A + 4B \le 160 & H \\
 & 35A + 20B \le 1190 & M \\
 & A, B \ge 0 &
\end{array}
$$

Vertices are where n non-degenerate constraints are tight

(equivalently, $m-n$ constraints are not tight)



Gives a set of candidate solutions and an optimality criterion:

Basic Feasible Solutions, Algebraically

slack form LP

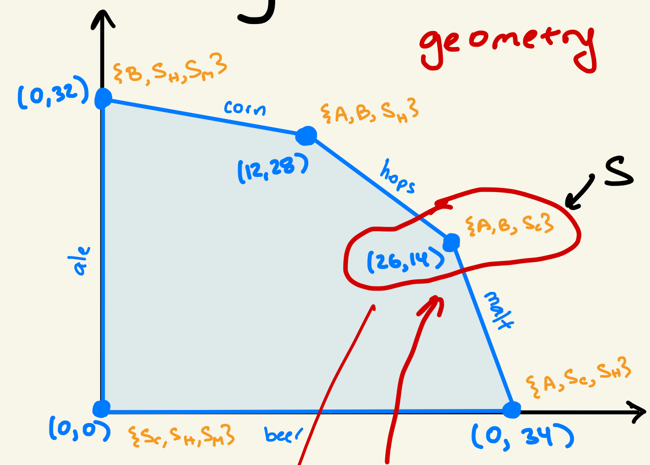
$$
\begin{aligned}
\max \quad & 13A + 23B \\
\text{s.t.} \quad & 5A + 15B + S_C = 480 \\
& 4A + 4B + S_H = 160 \\
& 35A + 20B + S_M = 1190 \\
& A, B, S_C, S_H, S_M \geq 0
\end{aligned}
$$

$\Downarrow$

$$
\begin{array}{ccccc}
(A) & (B) & (S_C) & (S_H) & (S_M) \\
\end{array}
$$
$$
\begin{bmatrix}
5 & 15 & 1 & 0 & 0 \\
4 & 4 & 0 & 1 & 0 \\
35 & 20 & 0 & 0 & 1
\end{bmatrix}
$$

constraint matrix A

more variables than constraints



basic feasible solution

(A)	(B)	(S_C)	(S_H)	(S_M)
?	?	?	0	0

$A_S^{-1} b$

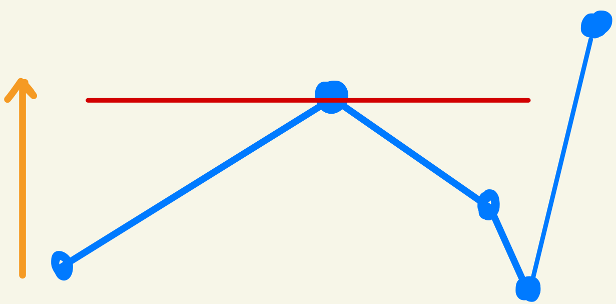
A restricted to columns in S

The Simplex Algorithm

Given an LP

$$\max_{x \in \mathbb{R}^n} \; c^T x$$
$$Ax = b$$
$$x \geq 0$$

$c \in \mathbb{R}^n$
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$



Algorithm

Find an initial BFS s
(specified by $m-n$ not tight constraints)

Until optimality:

  Find an adjacent BFS s'
  with higher objective

Thm: If you terminate, you are at a global optimum (by convexity)

The Simplex Algorithm

Given an LP

$$\max_{x \in \mathbb{R}^n} \quad c^T x$$
$$Ax = b$$
$$x \geq 0$$

$c \in \mathbb{R}^n$
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

Algorithm

Find an initial BFS s
(specified by $m-n$ tight constraints)

Until optimality:

- Find an adjacent BFS s' with higher objective

The Simplex Algorithm in Practice

Issues:
1. Choose a good pivoting rule
2. Avoid cycling (for degenerate LPs)
3. Maintain sparsity
4. Numerical stability
5. Preprocessing to reduce the size of the LP

Theory: Simplex might need exponentially many pivots

Practice: Can solve LPs with millions of variables / constraints (typically $\leq 2(n+m)$ pivots)