CS7800: Advanced Algorithms

Greedy Algs II

- Minimum spanning trees
- Classroom scheduling

Jonathan Ullman

09-16-2022

Network Design Problems

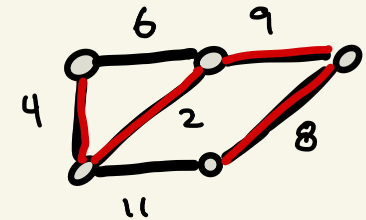
Given:

- A set of nodes V
- A set of potential edges E
- Costs $\{w_e\}$ for building each edge

$$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} G = (V, E, \{w_e\})$$

build a network that is well connected and cheap

Minimum Spanning Trees



Given:

- A set of nodes V
- A set of potential edges E
- Costs $\{w_e\}$ for building each edge
  ≥ 0

$G = (V, E, \{w_e\})$
≥ 0

build a network that is well connected and cheap

A subgraph $T = (V, E')$

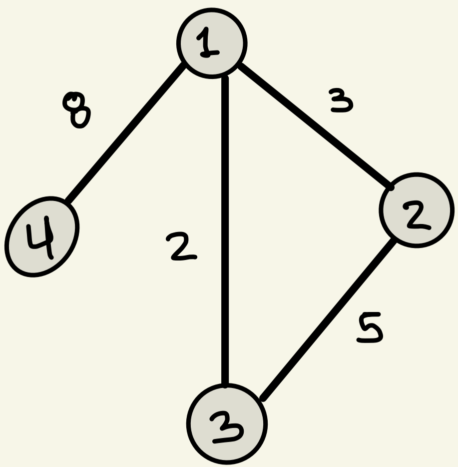
T is connected

minimize $\sum_{e \in T} w_e$

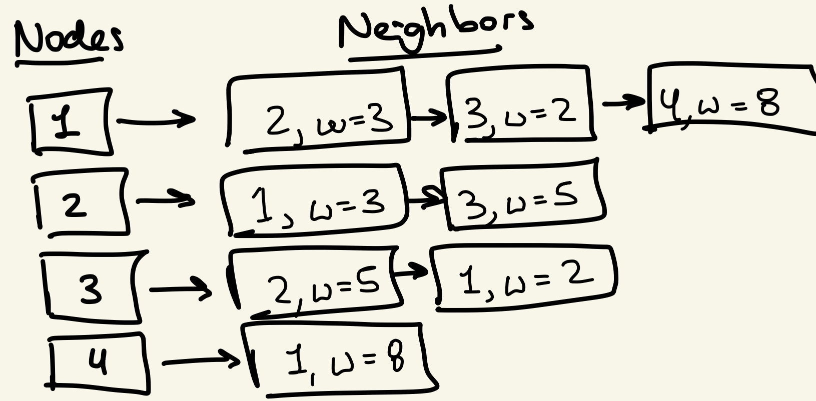
T is a tree

Brief Aside: Adjacency List Representation

Graph $G = (\overset{|V|=n}{V}, \overset{|E|=m}{E}, \{w_e\})$



Adjacency List of G



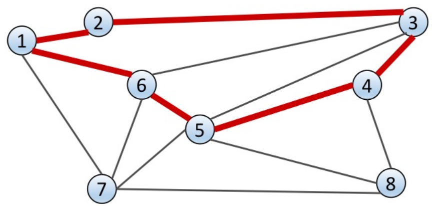
Size: $O(n+m)$

Time to List Neighbors of i : $O(\deg(i)+1)$

Time to List All Edges : $O(n+m)$

Cycles and Cuts

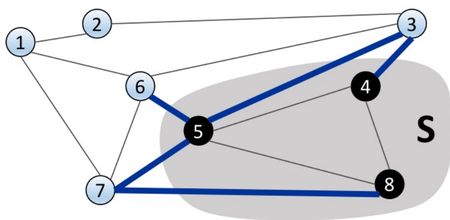
- **Cycle:** a set of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1)$



1, 2, 3, 4, 5, 6, 1

Cycle C = (1,2),(2,3),(3,4),(4,5),(5,6),(6,1)

- **Cut:** a partition of the nodes into $S, \bar{S}$



Cut S	=	{4, 5, 8}
Cutset	=	(5,6), (5,7), (3,4), (3,5), (7,8)

$$\text{Cutset}(S) = \{ e = (i,j) : i \in S, j \in \bar{S} \}$$

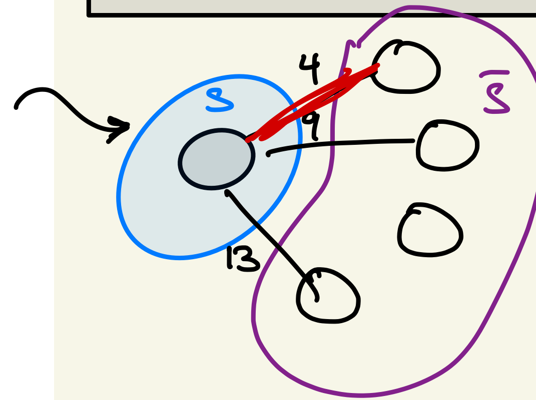
__Fact:__ Any cycle intersects any cut in an even number of edges.

Cycle and Cut Property of MST

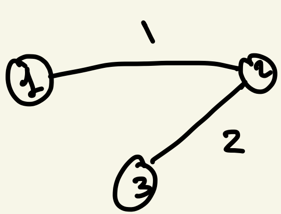
Assumption:

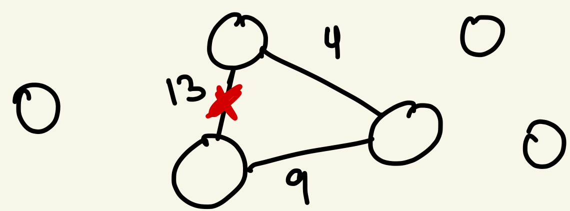
All we are distinct

- **Cut Property:** Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e
  - We call such an e a safe edge



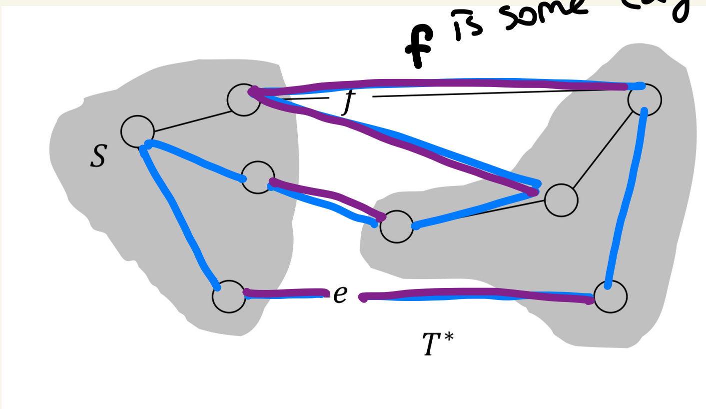
- **Cycle Property:** Let C be a cycle. Let f be the maximum weight edge in C. Then the MST T^* does not contain f.
  - We call such an f a useless edge





Proof of Cut Property

f is some edge on the cycle



- **Cut Property:** Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e
    - We call such an e a safe edge

- let T^* be the MST
- let $S, \bar{S}$ be a cut
- let e be the min wt edge in the cutset of S
- assume $e \notin T^*$ (for contradiction)
- adding e to T^* would create a cycle C containing e
- C intersects the cut in ≥ 2 places, let $f \neq e$ be one of them $w_f > w_e$
- consider $T' = T^* + \{e\} - \{f\}$ $\left| \begin{array}{l} ① \ wt(T') < wt(T^*) \\ ② \ T' \text{ is a spanning tree} \end{array} \right)$

The "Only MST Algorithm"

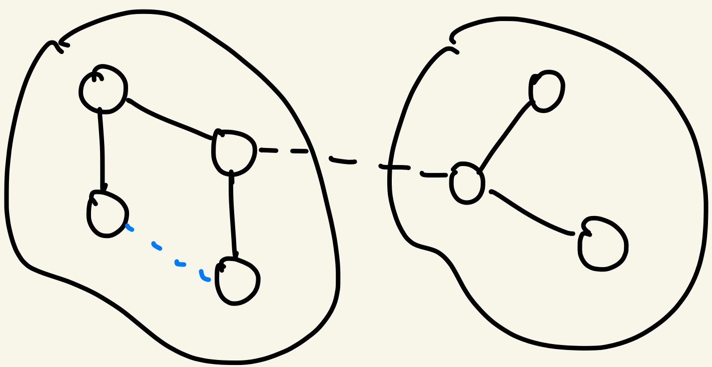
Let $T = (V, \emptyset)$

While T is not connected:

  add one or more new safe edges

Return T

Always terminates and outputs an MST no matter how we choose safe edges



Borůvka's Algorithm

Input: $G = (V, E, \{w_e\})$

Let $T = (V, \emptyset)$

count $\leftarrow$ CountLabel(T)

While count > 1

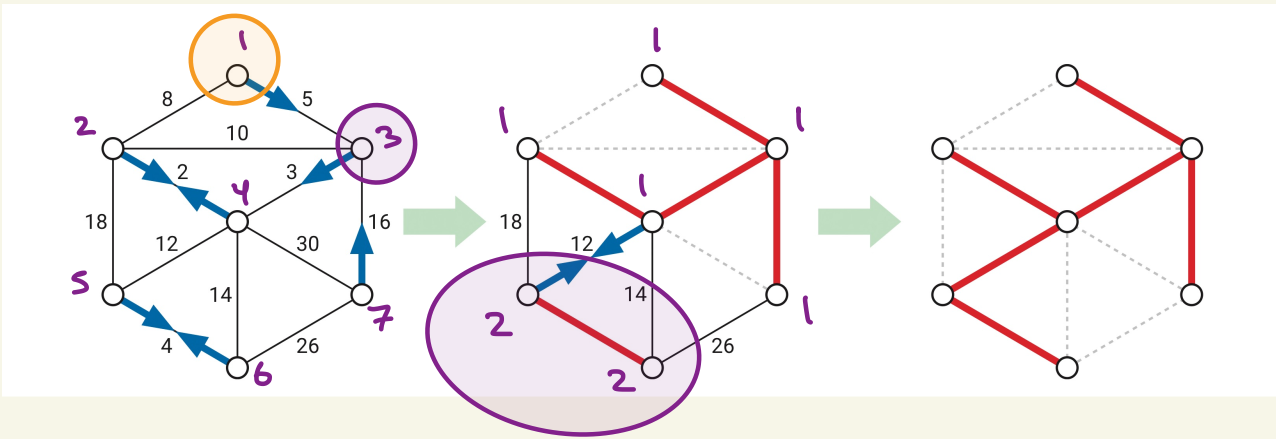
- AddSafeEdges(G, T, count)
- count $\leftarrow$ CountLabel(T)

Return T

Count connected components
Label nodes by component

Find the min-wt edges
leaving each connected component

Borůvka's Algorithm



Borůvka's Algorithm Correctness

Input: $G = (V, E, \{w_e\})$

Let $T = (V, \emptyset)$

count $\leftarrow$ CountLabel(T)

While count > 1

- AddSafeEdges(G, T, count)
- count $\leftarrow$ CountLabel(T)

Return T

Follows from the general template

Borůvka's Algorithm Running Time

```
Input: G = (V, E, {w_e})

Let T = (V, ∅)
 count ← CountLabel(T)
 While count > 1
     AddSafeEdges(G, T, count)
     count ← CountLabel(T)
Return T
```

How many iterations?

$O(\log n)$

How much time per iteration?

$O(m)$

Total Time: $O(m \log n)$

Borůvka's Algorithm Running Time

Input: $G = (V, E, \{w_e\})$

Let $T = (V, \emptyset)$

count $\leftarrow$ CountLabel(T)

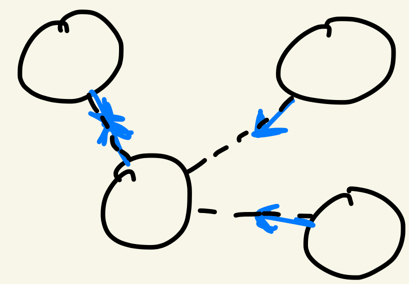
While count > 1

  Add Safe Edges(G, T, count)

  count $\leftarrow$ CountLabel(T)

Return T

How many iterations?



- if I have count components I will add $\geq \frac{\text{count}}{2}$ edges
- every edge reduces count by 1
- next iteration there are at most $\frac{\text{count}}{2}$ components $\Rightarrow O(\log n)$

Borůvka's Algorithm Running Time

Input: $G = (V, E, \{w_e\})$

Let $T = (V, \emptyset)$

count ← CountLabel(T)

While count > 1

  Add Safe Edges(G, T, count)

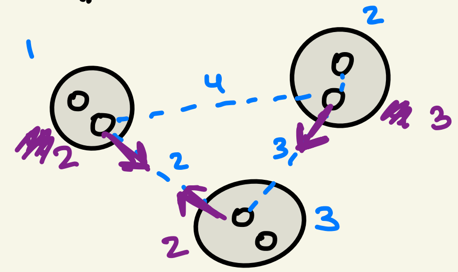
  count ← CountLabel(T)

Return T

How much time per iteration?

array for component labels

1	2	3	4	5	6
1	2	2	3	3	1

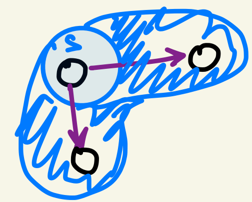


loop over edges

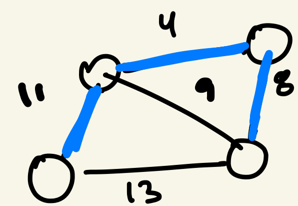
mantain cheapest edge so far

time : $O(n+m)$

Other MST Algorithms



- Prim:
  - Maintain a component S
  - Initially $S = \{v_1\}$
  - Add the safe edge for S until $S = V$



- Kruskal:
  - Sort edges
  - For each edge add it to T if it merges two components into one.

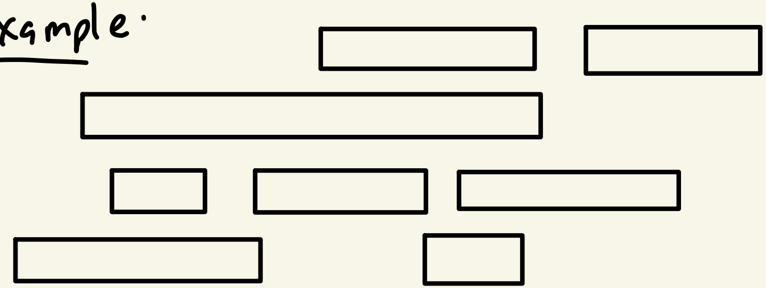
One More Greedy Proof Technique

<u>Input</u>: n "classes" $[s_i, f_i]$

<u>Output</u>: A "room" for each class so that

1. No room has two overlapping classes
2. The total number of rooms is as small as possible

<u>Example</u>:



4 classrooms

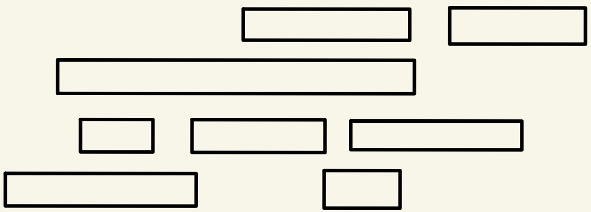
A Greedy Algorithm

Sort intervals by start

$s_1 \leq s_2 \leq \ldots \leq s_n$

For $i = 1, \ldots, n$:

assign class i to the lowest numbered empty room



Proof of Correctness: Duality

$$\overbrace{\underbrace{TTTT\ldots T}_{x}\,H\;TTH\;TH\;\ldots\;TH}^{y}$$

$x+1$ heads